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*Final Report on*

# *The Effect of Nonlinearities on Flexible Structures*

**Grant No. AFOSR-86-0090**

*by*


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February 19, 1989

# *The Effect of Nonlinearities on Flexible Structures*

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## *Abstract*

The project is a theoretical and experimental investigation into the influence of nonlinearities on flexible structures in the presence of multifrequency parametric and external excitations having independent frequencies and phases and arbitrary amplitudes. The nonlinearities and excitations may appear in the governing equations, or the boundary conditions, or both. The study focused on resonance conditions that produce large and possibly damaging motions. Special attention was given to modal coupling and exchanges of energy. We classified the important resonances and their interactions and devised experiments illustrating the phenomena.

## *I. Summary of Findings*

The reported research investigated theoretically and experimentally the response of flexible structural systems to multifrequency excitations. The excitations may be external (appear as inhomogeneous terms in the governing equations and boundary conditions) or parametric (appear as time-dependent coefficients in the governing equations and boundary conditions). The sources of the nonlinearities in the governing equations may be geometric, or inertial, or material, or any combination. The geometric nonlinearity stems from nonlinear strain-displacement relations (e.g., mid-plane stretching, large curvatures of structural elements, and large rotations of elements), the inertial nonlinearity may be caused by the presence of concentrated or distributed masses (in a Lagrangian formulation, the kinetic energy is a function of the generalized coordinates as well as their rates), and the material nonlinearity occurs when the stresses are nonlinear functions of the strains. The nonlinearities may appear in the governing partial differential equations, or the boundary conditions, or both. The form of the nonlinearity appearing in the equations and boundary conditions depends on the coordinate system being used.

## 1.1 Parametric Resonances

Modeling a system that is subjected to a parametric excitation by linear equations and boundary conditions is unrealistic if the parametric excitation leads to an instability because such a model predicts unlimited amplitudes. The predicted growth of the response is exponential. Consequently, a more realistic model includes nonlinear terms which act as limiters of the predicted response. Moreover, the linear model may predict a parametric stability (i.e., decaying response), when the actual response may not decay under certain conditions. In this case, the parametric excitation produces a so-called subcritical instability that is only predictable by including nonlinear terms.

Zavodney and Nayfeh studied the nonlinear response of a slender continuous beam with a concentrated mass located between the ends. The support undergoes a harmonic motion as shown in Figure 1. The motion is governed by the following nonlinear integro-partial-differential equation:

$$\begin{aligned} &[\rho + m\delta(s-d)]\ddot{v} + c\dot{v} + EI\left\{v^{iv} + [v'(v'v'')']'\right\} \\ &\quad - \frac{\partial}{\partial s}[J\delta(s-d)(v')_{tt}] - \frac{\partial}{\partial s}(Nv') = 0 \end{aligned} \tag{1}$$

where

$$\begin{aligned} N = &\frac{1}{2}\rho\int_s^L\left[\int_0^\xi (v'^2)_{tt}dn\right]d\xi - \frac{1}{2}m\int_s^L \delta(\xi-d)\left[\int_0^\xi (v'^2)_{tt}d\eta\right]d\xi \\ &+ m(\ddot{z}-g)\int_s^L \delta(\xi-d)d\xi + \rho L\left(1-\frac{s}{L}\right)(\ddot{z}-g) \end{aligned} \tag{2}$$

the prime indicates the derivative with respect to the spatial argument, and the overdot indicates the derivative with respect to the time t. The boundary conditions are

$$v(0,t) = 0,\ v'(0,t) = 0,\ v''(L,t) = 0,\ v'''(L,t) = 0 \tag{3}$$

Using a combination of the Galerkin procedure and the method of multiple scales, we determined a first-order uniform expansion when $z(t) = f\cos\Omega t$. A typical frequency-response curve is shown in Figure 2. Here, $\phi = \Omega/\omega_0$, where $\omega_0$ is the linear natural frequency of the lowest mode of vibration, and the dashed curves represent unstable solutions. Figure 3, which corresponds to region II of Figure 2, shows that parametric vibrations exist only when the excitation amplitude exceeds a threshold value. When the frequency of the excitation is increased to region III, the frequency-response curve is multivalued, resulting in a subcritical instability. This is shown in Figure 4.

We conducted experiments for composite and metallic beams. For the composite beam, we fabricated a symmetrical $0-90-90-0°$ 4-ply graphite-epoxy composite plate 0.022 inches thick and cut it into strips one-half inch wide. The frequency-response curves of the composite beam for three levels of excitation amplitude are shown in Figure 5. We observe that the general behavior is as predicted by the theory. There is, however, a maximum frequency at which point further increases in the frequency cause a jump down to the lower branch. We also note the appearance of chaotic behavior for the largest amplitude response; it was preceded by a modulation in the amplitude. We observe a penetration of a stable parametric resonance. Inside this "unstable" region small disturbances decayed and large disturbances grew, but as the region was penetrated, the disturbances that decayed became smaller and smaller until the trivial solution become unstable to all disturbances. We also see the lower branch lifting off the frequency axis as the frequency is decreased from above. This behavior was not predicted and appears to be intensified due to the higher level and nonlinear nature of the damping present in the composite beam.

The nature of the parametric resonance for $\phi = 2.000$ and for a table acceleration level of 1.00g is shown in Figure 6(a). When the excitation frequency is increased to $\phi = 2.013$ and the model is released from rest, the lower branch attracts the response, as shown in Figure 6(b). After the system achieves steady state, it is disturbed, and we note that the disturbance causes the system to jump up to the large amplitude response. Here the system modulates and does not achieve a constant steady-state amplitude.

For the metallic beam experiments, we fabricated a flexible steel beam, instrumented it with a strain gage, and fitted it with a small mass. The frequency-response curve is shown in Figure 7. We again note a penetration of a stable trivial solution into the region predicted to be unstable by the theory, and we note a modulation in the large steady-state amplitude just before it jumps down. The amplitude response for $\phi = 2.000$ is shown in Figure 8. These results also show remarkable agreement with the theory (see Figure 3).

## 1.2 Single-Degree-of-Freedom Systems

Including nonlinearity in the governing equations can lead to the prediction of a whole range of phenomena that are not found in linear mathematical models, but are found in the actual structures. In single-degree-of-freedom systems subject to stationary excitations, these phenomena include multiple solutions, jumps, limit cycles, frequency entrainment, natural frequency shift, and subharmonic, superharmonic, combination, and ultrasubharmonic resonances, period-multiplying bifurcations, and chaotic motions.

Nonlinear mathematical models predict that the devastating effects of a one-harmonic load having a frequency near the natural frequency might be lowered to a tolerable level by simply adding one or more nonresonant harmonic loads, which can shift the natural frequency [1]. They also predict that the large response can be significantly reduced by simply adding other resonant loads having the proper amplitudes and phases [3]. For example, the large response of single-degree-of-freedom system with quadratic and cubic nonlinearities to a parametric excitation only is shown in Figure 9b, the response of the same system to a subharmonic excitation only is shown in Figure 9a, the response to the combined excitations when the relative phase angle $\tau = 0$, is shown in Figure 9c, and the response to both excitations when $\tau = \pi$ is shown in Figure 9d. Figure 9 demonstrates the dramatic quenching of the response of the system to a principal parametric excitation by the addition of a subharmonic excitation, which by itself does not excite the system [3]. A third prediction is that when the sum of three frequencies in the load equals or nearly equals the natural frequency, the system can experience a combination-resonant response in which the peak amplitudes are several times larger than those predicted by linear theory [1]. An example is shown in Figure 10. A fourth prediction is that variations in the amplitude and phase of an excitation can strongly influence the response of nonlinear systems [1]. In Figure 11, we compare the stationary and nonstationary responses of a system to a primary excitation. It follows from this figure that the response depends strongly on the rate $r$ of the increase or decrease of the amplitude $k$ of the excitation. Thus, the present example clearly illustrates the need to consider nonstationary effects in determining the response of structural elements.

## 1.3 Multi-Degree-of-Freedom Systems

Besides the above phenomena, the response of multidegree-of-freedom systems to periodic excitations can exhibit combinational resonances and what is generally referred to as modal interactions; the latter may provide a coupling or an energy exchange among the system's modes [1,4]. This coupling can dominate the response of systems having some modes that are involved in internal or autoparametric resonances (i.e., the linear natural frequencies $\omega_i$ are commensurable or nearly commensurable).

The types of internal resonances depend on the degree of the nonlinearity. When the nonlinearity is cubic, internal resonance may occur if $\omega_n \approx \omega_m$, $\omega_n \approx 3\omega_m$, $\omega_n \approx |\pm 2\omega_m \pm \omega_k|$, or $\omega_n \approx |\pm \omega_m \pm \omega_k \pm \omega_\ell|$. When the nonlinearity is quadratic, besides the above resonances, internal resonances may occur if

$\omega_n \approx 2\omega_m$ or $\omega_n \approx \omega_m + \omega_k$. In contrast with single-degree-of-freedom systems in which combination resonance occur only if the excitation involves multiple frequencies, combination resonances may be exhibited in the response of multidegree-of-freedom systems to a single-harmonic excitation of frequency $\Omega$. The type of combination resonance that can be excited depends on the degree of the nonlinearity. For a cubic nonlinearity, combination resonances may have one or more of the following forms: $\Omega \approx | \pm \omega_n \pm \omega_m \pm \omega_k |$, $\Omega \approx | \pm \omega_n \pm 2\omega_m |$, or $\Omega \approx \frac{1}{2} | \pm \omega_n \pm \omega_m |$. For a quadratic nonlinearity, combination resonances may have the form $\Omega \approx | \pm \omega_n \pm \omega_m |$, besides the above forms

Internal resonances are responsible for many interesting, unusual, and dangerous phenomena. For example, they are responsible for the instability of the planar motions of a string or a symmetric beam resulting from a harmonic planar force [1,5,6]. Experiments show, and the nonlinear analysis predicts, that the response of a string or a symmetric beam to a plane harmonic excitation is planar provided the excitation amplitude is below a critical value. Above this critical value, the planar motion becomes unstable and gives way to a nonplanar, whirling motion. The whirling motion is a direct consequence of the fact that the natural frequency for motion in the plane of the excitation is the same as the natural frequency for motion in the plane perpendicular to the plane of the excitation.

### 1.4 One-to-One Autoparametric Resonances

As an example, Nayfeh and Pai [5] theoretically investigated the planar and nonplanar responses of a fixed-free beam to a principal parametric excitation. The beam is assumed to undergo flexure about two principal axes and torsion, as shown in Figure 12. The equations governing the parametric vibration of the system shown in Figure 12 are

$$\ddot{v} + c\dot{v} + \beta_y v^{iv} = (1 - \beta_y)\left[ w'' \int_1^s v''w''ds - w''' \int_0^s v''w'ds \right]'$$
$$- \frac{(1 - \beta_y)^2}{\beta_y} \left[ w'' \int_0^s \int_1^s v''w''dsds \right]'' - \beta_y [v'(v'v'' + w'w'')']' \tag{4}$$
$$- \frac{1}{2} \left\{ v' \int_1^2 \left[ \int_0^s (v'^2 + w'^2) ds \right]^{\cdot\cdot} ds \right\}' - [v''(s - 1) + v'] B\Omega^2 \cos(\Omega t)$$

$$\ddot{w} + c\dot{w} + w^{iv} = -(1 - \beta_y)\left[ v'' \int_1^s v''w''ds - v''' \int_0^s w''v'ds \right]'$$
$$- \frac{(1 - \beta_y)^2}{\beta_y} \left[ v'' \int_0^s \int_1^s v''w''dsds \right]'' - [w'(v'v'' + w'w'')']' \tag{5}$$
$$- \frac{1}{2} \left[ w' \int_1^s \left[ \int_0^s (v'^2 + w'^2) ds \right]^{\cdot\cdot} ds \right]' - \left[ - [w''(s - 1) + w'] B\Omega^2 \cos(\Omega t) \right]$$

and the boundary conditions are

$$v = w = v' = w' = 0 \text{ at } s = 0 \tag{6}$$

$$v'' = w'' = v''' = w''' = 0 \text{ at } s = 1 \tag{7}$$

Equations (4) and (5) contain cubic nonlinearities due to curvature and inertia. Nayfeh and Pai considered two uniform beams with rectangular cross sections: one has an aspect ratio near unity, and the other has an aspect ratio near 6.27. In both cases, the beam possesses a one-to-one internal resonance with one of the natural flexural frequencies in one plane being approximately equal to one of the natural flexural frequencies in the second plane.

We used a combination of the Galerkin procedure and the method of multiple scales to construct a first-order uniform expansion for the interaction of the two resonant modes, obtaining four first-order nonlinear ordinary-differential equations governing the amplitudes and phases of the modes of vibration. Figure 13 shows the response curves of the first mode. It is well known that, for planar responses, if only in-plane disturbances are considered then the upper branch of $a_2$ will be stable. But this branch can be unstable to disturbances in the y-direction. We note that the planar response curves are bent to the right, which implies that the nonlinear geometric terms dominate the response because they have a hardening effect.

Neglecting the nonlinear geometric terms, we obtained the response curves shown in Figure 14. Comparing Figures 13 and 14 shows that neglecting the geometric nonlinearity yields frequency curves that are even qualitatively wrong. The nonlinearity changes from a hardening to a softening type. Moreover, nonplanar responses cannot be predicted without including the geometric nonlinearity.

As $\sigma$ decreases from a value larger than that corresponding to point B in Figure 13, the nonplanar fixed point loses stability with a complex conjugate pair of eigenvalues moving into the right-half plane. This corresponds to the extensively studied Hopf bifurcation. Based on the Hopf bifurcation theorem, one expects amplitude- and phase-modulated motions for values of $\sigma$ near $B$. We note that all the nonplanar response curves are bent to the right even for the second mode. This is not unexpected because as discussed earlier the nonlinear geometric terms control the nonplanar motion. For the first mode, the nonlinear geometric terms have a hardening effect, and hence the amplitudes of nonplanar motion are smaller than those of planar motion. On the other hand, for the second mode, the nonlinear geometric terms overcome the inertia terms and produce nonplanar motions that are larger than the planar motions. Using a Runge-Kutta routine to integrate the modulation equations in the region between the Hopf bifurcation points for a long period of time, we obtained the amplitude-modulation behavior shown in Figure 15. Figure 16 shows the projection of the attractor on the $a_2 - a_1$ plane. Since the amplitudes and phases are not constant but periodic with a period that is larger than that corresponding to free oscillations, the resulting motion is nonperiodic having two periods (i.e., motion on a torus). The spectrum of $a_1$ in Figure 17 shows that the fundamental dimensionless frequency of the attractor is approximately 0.154 and hence its dimensionless period is approximately 6.5. This motion can be better visualized by plotting the motion of the tip-end of the beam, as shown in Figure 18. This figure shows that the elliptical route keeps changing the lengths of axes and direction, and it also shows the twisting motion. Because of the nonlinear terms, the inertia force in the y-direction is not proportional to $v(s,t)$ and the inertia force in the z-direction is not proportional to $w(s,t)$, and hence the resultant inertia force is not parallel to the total displacement in the y-z plane and it induces a twisting moment on the beam. This is a whirling motion of the beating type.

### 1.5 Two-to-One Autoparametric Resonances

Two-to-one autoparametric or internal resonances (i.e., $\omega_2 \approx 2\omega_1$) are responsible for saturation in the response of systems with quadratic nonlinearities [1,4,7,8]. When the second mode is excited by a single-harmonic load of amplitude $F$ and frequency $\Omega \approx \omega_2$, one expects the second mode to be dominant; initially this is so. But as $F$ increases above a critical value $F_c$, the second mode becomes saturated and the additional energy spills over into the first mode (see Figure 19). The threshold value $F_c$ depends on the damping and detunings of the resonances and can be very very small. Thus, the region of validity of the linear solution can be very very small, and consequently, controls based on linear theory may lead to undesirable conditions. Saturation was predicted by the nonlinear analysis and has recently been observed experimentally in structures consisting of metallic beams and concentrated masses. An example of these structures is shown in Figure 20. It consists of two light-weight steel beams and two concentrated masses. The mass on the vertical beam can be moved up and down to adjust the natural frequencies. The first two natural frequencies are below 25 Hz, whereas the third natural frequency is above 100 Hz. The first two frequencies correspond to the lowest two flexural modes whereas the third frequency corresponds to the first torsional mode. Hence, at low excitation frequencies, this structure is essentially a two-degree-of-freedom structure. When the second natural frequency $\omega_2$ is approximately twice the first natural frequency $\omega_1$, we have found that nonlinear modal interactions exist between these modes. Fixing the excitation frequency $\Omega$ at a value near $\omega_2$ and slowly increasing the excitation amplitude F from zero, we have found that initially the amplitude $a_2$ of the second mode increases linearly with F while the amplitude

$a_1$ of the first mode remains zero, in accordance with linear theory. However, as F increases beyond a threshold value $F_c$, $a_2$ remains constant (saturates) and the extra input energy spills over into the first mode, which grows rapidly as F increases further. This is the saturation phenomenon. The response of the structure consists of a combination of the two flexural modes. For certain amplitudes and frequencies of the excitation, the response of the structure ceases to be periodic and becomes either two-period quasi-periodic or chaotic motion. However, in all cases, the motion continues to be planar consisting of the two lowest flexural modes.

In preparing for the Sixth Annual Review of the Center for Composite Structures and Materials on April 9-11, 1989, we thought to investigate whether composite structures exhibit the above complicated behaviors that we observed in the response of the metallic structures. To this end, we fabricated a composite plate 85 mili-inch thick from 7781/ 5245C glass/epoxy, 0°/90° woven fabric material by using the stacking sequence $[0°/90°/45°/-45_2°/45°/90°/0°]$. We cut the plate into 0.51″ wide strips. We used two composite strips of length 8″ and two concentrated masses to build a structure similar to that shown in Figure 21. We used an epoxy adhesive to bond the composite strips to perpendicular faces of the junction mass. We mounted the structure on the shaker and set the excitation level at about 60 mili g's and swept the frequency of excitation. To our surprise, the vertical beam went into out-of-plane (torsional) vibrations for certain excitation frequencies. Then we set the excitation frequency near 16.0 Hz and slowly increased the excitation amplitude. Initially, the response was planar. However, above a threshold excitation level, the vertical beam went into an out-of-plane (torsional) vibration. The bonding provided by the adhesive did not hold when the composite structure went into torsional motions. Therefore, we replaced the adhesive bonding with an L-shaped clamp made of aluminum, as shown in Figure 21.

To understand the physical mechanisms underlying these observations, we instrumented the composite model by mounting strain gages along the axes of the horizontal and vertical beams (referred to as Strain Gage H and

Strain Gage V in Figure 21) to measure the displacements due to flexural motions and strain gages on the top and bottom faces of the horizontal beam at 45° to its axis (referred to as Strain Gage II in Figure 21) to measure the displacements due to torsional motions. Also, we mounted an accelerometer on the shaker table to measure the excitation amplitude. Using a combination of free oscillations, random excitations, and sinusoidal sweeps, we found that the two lowest flexural frequencies to be 5.58 Hz and 16.59 Hz and the lowest torsional frequency to be 8.41 Hz. Thus, at low excitation frequencies, the model is essentially a three-degree-of-freedom system, in contrast with the metallic structure which is essentially a two-degree-of-freedom system. The reason for the low torsional frequency of the composite structure is the relative weakness in shear of composite laminates due to their low transverse shear moduli. Thus, the composite structure has a low torsional stiffness, which in turn implies that its torsional modes have lower natural frequencies than the corresponding modes in the metallic structure.

We note that the second flexural frequency is approximately twice the lowest torsional frequency and the second flexural frequency is approximately three times the first flexural frequency. Thus, the structure possesses two-to-one and three-to-one internal or autoparametric resonances. According to our previous theoretical and experimental findings, we expect the composite structure to exhibit nonlinear modal interactions. Because the two-to-one autoparametric resonance is due to quadratic nonlinearities and the three-to-one autoparametric resonance is due to cubic nonlinearities, the excitation levels needed to activate the former resonance are far below those needed to activate the second resonance. This is born out by our experiments.

We held the excitation frequency constant at 16.8 Hz, which is near the natural frequency of the second flexural mode, and slowly swept up and down the excitation amplitude. To distinguish linear from nonlinear motions and periodic from nonperiodic motions, we used the following schemes:

(a) visual observations of the model

(b) the cross-plots of the signals from the strain gages along the axes of the laminates

(c) the Poincáre maps obtained by using the excitation frequency as the clock frequency

(d) the FFT's of the accelerometer output and the strain gage signal mounted on the vertical beam.

The experimentally determined response amplitudes from the FFT signals are shown in Figure 22. The results obtained during the forward and reverse sweeps are marked by circles and triangles, respectively. The symbols $a_2^*$ and $a_t^*$ correspond to the amplitudes of the second flexural mode and the torsional mode, respectively. Initially, as the amplitude F of the excitation increases from zero, the amplitude $a_2^*$ of the second flexural mode increases linearly with F, whereas the amplitude $a_t^*$ of the torsional mode remains zero. As the excitation amplitude exceeds a threshold, which is very small (in this case about 11.0 mili g's), $a_2^*$ remains constant (saturates) and $a_t^*$ increases rapidly with F. Thus, the extra energy input to the second flexural mode spills over into the torsional mode whose amplitude can increase dramatically, with possible serious consequences to the integrity of the structure.

In another set of experiments, we found that the lowest two flexural frequencies of the structure are 5.45 Hz and 16.42 Hz and the first torsional frequency is 8.09 Hz. Again, for this configuration, two-to-one and three-to-one autoparametric resonances exist, and one expects strong modal interactions to exist between the second flexural mode and the first torsional mode. In this set, we held the excitation amplitude constant at 59.0 mili g's and slowly varied the excitation frequency around the second flexural frequency. In Figure 23, we show the spectra of the accelerometer and strain gage V signals as well as the cross-plots of the H and V signals at the excitation frequencies 16.10 Hz, 16.14 Hz, and 16.30 Hz. It follows from Figure 23a that at 16.10 Hz, the response is linear. The output is at the same frequency as the input and the cross-plot of the H and V signals is a simple limit cycle, indicating the presence of a single frequency in both the H and V signals. However, it follows from Figure 23b that at 16.14 Hz, the response is nonlinear and periodic. Now the output is at both the input frequency f and its one-half subharmonic $\frac{1}{2}f$. Moreover, the cross-plot of the H and V signals is an eight-shaped pattern, indicating the presence of two frequencies that are in the ratio of two-to-one in the signals. The subharmonic signal corresponds to the torsional mode, which was visually apparent at this frequency. It follows from Figure 23c that at 16.30 Hz, the response is nonlinear and nonperiodic. In addition to the presence of the frequencies f and $\frac{1}{2}f$ in the output, there are also side bands, which indicates the presence of two periods. If these periods are not commensurable, one talks of a two-period quasi-periodic, or amplitude- and phase-modulated, motion. The cross plot also indicates that the response is nonperiodic because the eight-shaped pattern continues to evolve. Furthermore, the torsional motion was clearly visible in the motion of the model.

In our frequency sweeps, we found ranges of frequencies where the response of the model is chaotic. An example of a chaotic response is shown in Figure 24 at the excitation frequency 16.325 Hz. The upper plot is the FFT of the V signal obtained by using a zoom span mode around 16.325 Hz whereas the lower plot is the FFT of the V signal obtained by using a base band mode. These spectra show broadband characters around the frequencies f and $\frac{1}{2}f$, indicating that the motion is a chaotically modulated, combined flexural-torsional motion. The Poincáre maps for a periodically modulated response (at 16.30 Hz) and a chaotically modulated response (at 16.325 Hz) are shown in Figure 25.

In summary, the dynamic behavior of the composite structure is both qualitatively and quantitatively different from that of the metallic structure. The low transverse shear moduli of composite laminates result in low torsional stiffnesses, which in turn result in low natural frequencies in torsion. Consequently, modal interactions between the flexural and torsional modes may occur, resulting in out-of-plane motions even when the structure is driven by in-plane excitations. This behavior may have serious implications for the integrity of composite structural elements and should be factored into their design.

## *II. Degrees Granted*

1. Ph.D. - Lawrence D. Zavodney, 1987 "A Theoretical and Experimental Investigation of Parametrically Excited Nonlinear Mechanical Systems".

2. Senior Project - Harold L. Neal, 1988 "Nonstationary Response of a Nonlinear System to Nonperiodic Parametric Excitations with Varying Frequency".

3. Ph.D. - Samir J. Serhan, 1989 "Response of Nonlinear Systems to Deterministic and Random Excitations"

4. Ph.D. - Raouf A. Raouf, 1989 "Modal Interactions in Shell Dynamics"

5. Ph.D. - Nestor E. Sanchez, 1989 "Stability of Nonlinear Oscillatory Systems with Application to Ship Dynamics"

## *III. Publications*

1. **Nayfeh, A. H., "Parametric Excitation of Two Internally Resonant Oscillators", Journal of Sound and Vibration, Vol. 119(1), 1987, pp. 95-109.**

   **Proceedings of Invited Lectures and Short Communications of the XI, International Conference on Nonlinear Oscillations, Budapest, Hungary, August 17-23, 1987, pp. 181-188.**

   The response of two-degree-of-freedom systems with quadratic nonlinearities to a principal parametric resonance in the presence of two-to-one internal resonances is investigated. The method of multiple scales is used to construct a first-order uniform expansion yielding four first-order nonlinear ordinary differential (averaged) equations governing the modulation of the amplitudes and the phases of the two modes. These equations are used to determine steady state responses and their stability. When the higher mode is excited by a principal parametric resonance, the nontrivial steady state value of its amplitude is a constant that is independent of the excitation amplitude, whereas the amplitude of the lower mode, which is indirectly excited through the internal resonance, increases with the amplitude of the excitation. However, in addition to Poincaré-type bifurcations, this response exhibits a Hopf bifurcation leading to amplitude- and phase-modulated motions. When the lower mode is excited by a principal parametric resonance, the averaged equations exhibit both Poincaré and Hopf bifurcations. In some intervals of the parameters, the periodic solutions of the averaged equations, in the latter case, experience period-doubling bifurcations, leading to chaos.

2. **A. H. Nayfeh and A. A. Khdeir, "Rolling of Ships in Large-Amplitude Waves," Dynamical Systems Approaches to Nonlinear Problems in Methods for the Analysis of Nonlinear Dynamics, New England College, Henniker, New Hampshire, June 8-13, 1986; also edited by Fathi M. A. Salam and Mark Levin, SIAM, 1988, pp. 290-303.**

A second-order approximate solution is presented for the nonlinear rolling response of ships in regular beams waves. A Floquet analysis is used to predict the stability of limit-cycle responses. The perturbation results are compared with solutions obtained by numerical integration of the nonlinear governing roll equation. The results show that the first-order perturbation expansion may be inadequate for predicting the peak roll angle and its corresponding frequency. On the other hand, the peak of the stable roll angle and corresponding frequency predicted by the second-order expansion are found to be in good agreement with the numerical simulation. Moreover, the perturbation expansion predicts fairly well the start of period multiplying bifurcations that lead to chaos. Biased ships are found to be more susceptible to period multiplying bifurcations and chaos than unbiased ships.

3. **Zavodney, L. D. and Nayfeh, A. H., "The Response of a Single-Degree-of-Freedom System with Quadratic and Cubic Non-Linearities to a Fundamental Parametric Resonance," Journal of Sound and Vibration, Vol. 120(1), 1988, pp. 63-93.**

The response of a one-degree-of-freedom system with quadratic and cubic nonlinearities to a fundamental harmonic parametric excitation is investigated. The method of multiple scales is used to determine the equations that describe to second order the modulation of the amplitude and phase with time about one of the foci. These equations are used to determine the fixed points and their stability. The perturbation results are verified by integrating the governing equation using a digital computer and an analogue computer. For small excitation amplitudes, the analytical results are in excellent agreement with the numerical solutions. As the amplitude of the excitation increases, the accuracy of the perturbation solution deteriorates, as expected. The large responses are investigated by using both a digital and an analogue computer. The cases of single- and double-well potentials are investigated. Systems with double-well potentials exhibit complicated dynamic behaviors including period multiplying and demultiplying bifurcations and chaos. Long-time histories, phase planes, Poincaré maps, and spectra of the responses are presented.

4. **Nayfeh, A. H. and Zavodney, L. D., "Experimental Observation of Amplitude- and Phase-Modulated Responses of Two Internally Coupled Oscillators to a Harmonic Excitation," Journal of Applied Mechanics, Vol. 110, 1988, pp. 706-710.**

An experiment is performed on a two-degree-of-freedom mechanical system having quadratic nonlinearities and linear natural frequencies $\omega_1$ and $\omega_2$ approximately in the ratio of two-to-one (i.e., $\omega_2 \approx 2\omega_1$). When the lower mode is excited by a harmonic excitation whose frequency $\Omega$ is nearly equal to $\omega_1$, amplitude- and phase-modulated responses of the system have been observed for a range of the excitation frequency $\Omega$, in qualitative agreement with the results of a second-order perturbation theory.

5. **Nayfeh, A. H. and Asfar, K. R., "Non-Stationary Parametric Oscillations," Journal of Sound and Vibration, Vol. 124(3), 1988, pp. 529-537.**

The response of a single-degree-of-freedom system with cubic nonlinearity to a nonstationary principal parametric excitation is investigated. The method of multiple scales is used to derive two first-order ordinary-differential equations for the evolution of the amplitude and phase of the response. The evolution equations are numerically integrated for various sweeping rates of the amplitude and frequency of the excitation. The results show that the nonstationary response penetrates the instability regions and the higher the sweeping rate is the deeper the penetration is.

6. **Nayfeh, A. H. and Pai, P. F., "Non-Linear Non-Planar Parametric Responses of an Inextensional Beam," International Journal of Non-Linear Mechanics, Vol. 24(2), 1989, pp. 139-158.**

The nonlinear integro-differential equations of motion for an inextensional beam are used to investigate the planar and nonplanar responses of a fixed-free beam to a principal parametric excitation. The beam is assumed to undergo flexure about two principal axes and torsion. The equations contain cubic nonlinearities due to curvature and inertia. Two uniform beams with rectangular cross sections are considered: one has an aspect ratio near unity, and the other has an aspect ratio near 6.27. In both cases, the beam possesses a one-to-one internal resonance with one of the natural flexural frequencies in one plane being approximately equal to one of the natural flexural frequencies in the second plane. A combination of the Galerkin procedure and the method of multiple scales is used to construct a first-order uniform expansion for the interaction of the two resonant modes, yielding four first-order nonlinear ordinary-differential equations governing the amplitudes and phases of the modes of vibration. The results show that the nonlinear inertia terms produce a softening effect and play a significant role in the planar responses of high-frequency modes. On the other hand, the nonlinear geometric terms produce a hardening effect and dominate the planar responses of low-frequency modes and nonplanar responses for all modes. If the nonlinear geometric terms were not included in the governing equations, then nonplanar responses would not be predicted. For some range of parameters, Hopf bifurcations exist and the response consists of amplitude- and phase-modulated or chaotic motions.

7. **Zavodney, L. D., Nayfeh, A. H., and Sanchez, N. E., "The Response of a Single-Degree-of-Freedom System with Quadratic and Cubic Nonlinearities to a Principal Parametric Resonance," Journal of Sound and Vibration, Vol. 129(3), 1989, pp. 417-442.**

   The response of a one-degree-of-freedom system with quadratic and cubic nonlinearities to a principal parametric resonance is investigated. The method of multiple scales is used to determine the equations that describe to second order the modulation of the amplitude and phase with time about one of the foci. These equations are used to determine the fixed points and their stability. The perturbation results are verified by integrating the governing equation using a digital computer and an analogue computer. For small excitation amplitudes, the analytical results are in excellent agreement with the numerical solutions. The large amplitude responses are investigated using both a digital and an analogue computer. The cases of single- and double-well potentials are investigated. Systems with double-well potentials exhibit complicated dynamic behaviors including period-multiplying and demultiplying bifurcations and chaos. In some cases, a bifurcated response coexists with another periodic attractor, and a chaotic attractor coexists with a periodic attractor. Long-time histories, phase planes, Poincaré maps, fractal basin maps, and spectral of the response are presented. A bifurcation diagram of many solutions in the excitation amplitude-excitation frequency plane is also presented.

8. **Zavodney, L. D. and Nayfeh, A. H., "The Non-Linear Response of a Slender Beam Carrying a Lumped Mass to a Principal Parametric Excitation: Theory and Experiment," International Journal of Non-Linear Mechanics, Vol. 24(2), 1989, pp. 105-125.**

The non-linear response of a slender cantilever beam carrying a lumped mass to a principal parametric base excitation is investigated theoretically and experimentally. The Euler-Bernoulli theory for a slender beam is used to derive the governing non-linear partial differential equation for an arbitrary position of the lumped mass. The non-linear terms arising from inertia, curvature and axial displacement caused by large transverse deflections are retained up to third order. The linear eigenvalues and eigenfunctions are determined. The governing equation is discretized by Galerkin's method, and the coefficients of the temporal equation-composed of integral representations of the eigenfunctions and their derivatives-are computed using the linear eigenfunctions. The method of multiple scales is used to determine an approximate solution of the temporal equation for the case of a single mode. Experiments were performed on metallic beams and later on composite beams because all of the metallic beams failed prematurely due to the very large response amplitudes. The results of the experiment show very good qualitative agreement with the theory.

9. **Nayfeh, A. H. and Balachandran, B., "Modal Interactions in Dynamical and Structural Systems," Applied Mechanics Reviews, Vol. 42(11), 1989, pp. 175-201.**

We review theoretical and experimental studies of the influence of modal interactions on the nonlinear response of harmonically excited structural and dynamical systems. In particular, we discuss the response of pendulums, ships, rings, shells, arches, beam structures, surface waves, and the similarities in the qualitative behavior of these systems. The systems are characterized by quadratic nonlinearities which may lead to two-to-one and combination autoparametric resonances. These resonances give rise to a coupling between the modes involved in the resonance leading to nonlinear periodic, quasi-periodic, and chaotic motions.

10. **Neal, H. L. and Nayfeh, A. H., "Response of a Single-Degree-of-Freedom System to a Nonstationary Principal Parametric Excitation," International Journal of Non-Linear Mechanics, November 1989.**

We examine the nonstationary response of a one-degree-of-freedom nonlinear system to a nonperiodic parametric excitation with varying frequency. We use the method of multiple scales to obtain equations governing the stationary and nonstationary responses of the system, and we analyze the stability of the stationary responses. The response displays several phenomena, including penetration of the trivial response into the unstable trivial region, oscillation of the response about the nontrivial stationary solution, convergence of the nonstationary response to the stationary solution, lingering of the nontrivial response into the stable trivial region, and rebounding of the nontrivial response. These phenomena are affected by the sweep rate, the initial conditions, and the system parameters. We use digital and analog computers to solve the original governing differential equation. The results of the simulations agree with each other and with those obtained by using the method of multiple scales.

11. **Nayfeh, A. H., Balachandran, B., Colbert, M. A., and Nayfeh, M. A., "An Experimental Investigation of Complicated Responses of a Two-Degree-of-Freedom Structure," Journal of Applied Mechanics, Vol. 56(4), 1989, pp. 960-967.**

Recent theoretical studies indicate that whereas large excitation amplitudes are needed to produce chaotic motions in single-degree-of-freedom systems, extremely small excitation levels can produce chaotic motions in multi-degree-of-freedom systems if they possess autoparametric resonances. To verify these results, we conducted an experimental study of the response of a two-degree-of-freedom structure with quadratic nonlinearities and a two-to-one internal resonance to a primary resonant excitation of the second mode. The responses were analyzed using hardware and software developed for performing time-dependent modal decomposition. We observed periodic, quasi-periodic, and chaotic responses, as predicted by theory. Conditions were found under which extremely small exciiation levels produced chaotic motions.

12. **Sanchez, N. E. and Nayfeh, A. H., Prediction of Bifurcations in a Parametrically Excited Duffing Oscillator," International Journal of Non-Linear Mechanics, 1990.**

The instability regions of the response of a damped, softening type Duffing oscillator to a parametric excitation are determined via an algorithm that uses Floquet theory to evaluate the stability of second-order approximate analytical solutions in the neighborhood of the nonlinear resonances of the system. It is shown that identification of the locus of instabilities of the periodic approximate solutions in the amplitude-frequency parameter space provides valuable information on the overall dynamic behavior of the system. The predictions are verified by using analog- and digital-computer simulations, which exhibit chaos and unbounded motions among other behaviors.

13. **Nayfeh, A. H. and Serhan, S. J., "Response Statistics of Nonlinear Systems to Combined Deterministic and Random Excitations," International Journal of Non-Linear Mechanics, 1990.**

A second-order closure method is presented for determining the response of nonlinear systems to random excitations. The excitation is taken to be the sum of a deterministic harmonic component and a random component. The latter may be white noise or harmonic with separable nonstationary random amplitude and phase. The method of multiple scales is used to determine the equations describing the modulation of the amplitude and phase. Neglecting the third-order central moments, we use these equations to determine the stationary mean and mean-square response. The effect of the system parameters on the response statistics is investigated. The presence of the nonlinearity causes multi-valued regions where more than one mean-square value of the response is possible. The local stability of the stationary mean and mean-square responses is analyzed. Alternatively, assuming the random component of the response to be small compared with the mean response, we determine steady-state periodic responses to the deterministic part of the excitation. The effect of the random part of the excitation on the stable periodic responses is analyzed as a perturbation and a closed-form expression for the mean-square response is obtained. Away from the transition zone separating stable and unstable periodic responses, the results of these two approaches are in good agreement. Comparisons of the results of these methods with that obtained by the method of equivalent linearization are presented.

14. **Pai, P. F. and Nayfeh, A. H., "Nonlinear Nonplanar Oscillations of a Cantilever Beam Under Lateral Base Excitations," International Journal of Non-Linear Mechanics, 1990.**

The nonplanar responses of a cantilevered beam subject to lateral harmonic base-excitation is investigated using two nonlinear coupled integro-differential equations of motion. The equations contain cubic nonlinearities due to curvature and inertia. Two uniform beams with rectangular cross sections are considered: one has an aspect ratio near unity, and the other has an aspect ratio near 6.27. A combination of the Galerkin procedure and the method of multiple scales is used to construct a first-order uniform expansion for the case of a one-to-one internal resonance and a primary resonance. The results show that the nonlinear geometric terms are important for the responses of low-frequency modes because they produce hardening spring effects. On the other hand, the nonlinear inertia terms dominate the responses of high-frequency modes. We also obtain quantitative results for nonplanar motions and investigate their dynamic behavior. For different range of parameters, the nonplanar motions can be steady whirling motions, whirling motions of the beating type, or chaotic motions. Furthermore, we investigate the effects of damping.

15. **Nayfeh, A. H., Raouf, R. A., and Nayfeh, J. F., "Nonlinear Response of Infinitely Long Circular Cylindrical Shells to Subharmonic Radial Loads," Journal of Applied Mechanics, 1990.**

The method of multiple scales is used to analyze the nonlinear response of infinitely long circular cylindrical shells (thin circular rings) in the presence of a two-to-one internal (autoparametric) resonance to a subharmonic excitation of order one-half of the higher mode. Four autonomous first-order ordinary-differential equations are derived for the modulation of the amplitudes and phases of the interacting modes. These modulation equations are used to determine the fixed points and their stability. The fixed points correspond to periodic oscillations of the shell, whereas the limit-cycle solutions of the modulation equations correspond to amplitude- and phase-modulated oscillations of the shell. The first-response curves exhibit saturation, jumps, and Hopf bifurcations. Moreover, the frequency-response curves exhibit Hopf bifurcations. For certain parameters and excitation frequencies between the Hopf values, limit-cycle solutions of the modulation equations are found. As the excitation frequency changes, all limit cycles deform and lose stability through either pitchfork or cyclic-fork (saddle-node) bifurcations. Some of these saddle-node bifurcations cause a transition to chaos. The pitchfork bifurcations break the symmetry of the limit cycles.

16. **Nayfeh, A. H., Mook, D. T., and Nayfeh, J. F., "Some Aspects of Modal Interactions in the Response of Beams," AIAA Paper No. 87-0777, presented at the 28th Structures, Structural Dynamics and Materials Conference, Monterey, CA, April 6-8, 1987.**

Some aspects of modal interactions in the nonlinear response of hinged-clamped beams to a harmonic excitation are investigated. The analysis accounts for a static axial force, a restraining spring, and modal damping. For such a beam, the second natural frequency is approximately three times the first natural frequency, a condition of internal or autoparametric resonance. The method of multiple scales is used to determine the equations that describe the modulation of the amplitudes and phases with damping, primary resonance caused by the excitation, and nonlinearity including the autoparametric resonance. They fixed points of these equations and their stability are determined. The autoparametric resonance is found to produce a strong coupling of the modes involved. For some range of parameters, Hopf bifurcations exist. The first points lose their stability with the real part of a complex conjugate pair of eigenvalues changes sign from negative to positive. In these ranges, steady-state periodic solutions do not exist, contrary to results predicted by linear multi-mode analyses or nonlinear single-mode analyses. Instead, the energy is continuously exchanged between the modes involved in the autoparametric resonance. Moreover, for small damping, the response may experience period-multiplying bifurcations and chaos.

17. **Nayfeh, A. H., "Numerical-Perturbation Methods in Mechanics," Computers & Structures, Vol. 30, No. 2, 1988, pp. 185-204.**

In many nonlinear problems in mechanics, the responses are so complicated (jumps, period-multiplying bifurcations, chaos, saturation) that it is impractical if not impossible to determine their salient features by using a purely numerical technique. For weakly nonlinear systems, perturbation techniques can be used quite effectively. However, purely analytical techniques are limited to systems with simple boundaries and composition. These limitations can be removed by combining analytical and numerical techniques. These points are illustrated by examples drawn from structural vibrations, sloshing of liquids in containers, and nonlinear stability of boundary layers. The combination of analytical and numerical techniques also can be very useful for treating linear wave propagation in nonhomogeneous media. The procedure is illustrated by an example: intensification and refraction of acoustic signals in partially choked converging-diverging ducts.

18. **Nayfeh, A. H., "Application of the Method of Multiple Scales to Nonlinearly Coupled Oscillators," Chapter III, from the Lasers, Molecules and Methods, Advances in Chemical Physics Volume LXXIII, edited by**

**Hirschfelder, Wyatt and Coaison, John Wiley & Sons, Inc., NY, 1989, pp. 137-196.**

In this chapter, we investigate the response of two-degree-of-freedom systems with quadratic nonlinearities to parametric and external resonant excitations in the presence of two-to-one internal (autoparametric) resonances. We use the method of multiple scales to construct a first-order uniform expansion yielding four first-order nonlinear ordinary differential (averaged) equations governing the modulation of the amplitudes and the phases of the two modes. These equations are used to determine periodic responses and their stability. The autoparametric resonance produces a strong coupling of the modes involved. For some range of parameters, Hopf bifurcations exist. The fixed points of the averaged equations lose their stability when the real part of a complex-conjugate pair of eigenvalues changes sign from negative to positive. In these ranges, steady-state periodic solutions do not exist. Instead, the response consists of amplitude- and phase-modulated motion, and for small damping it may experience period multiplying bifurcations and chaos.

19. **Nayfeh, A. H., El-Zein, S. M. and Nayfeh, J. F., "Nonlinear Oscillations of Composite Plates Using Perturbation Techniques," Proceedings of the 4th Technical Conference on Composite Materials, VPI&SU, Blacksburg, VA, October 3-6, 1989, pp. 570-579.**

The method of multiple scales is used to analyze the nonlinear response of an antisymmetric cross-ply laminate to a harmonic load. The classical lamination theory, including rotary inertia, is used. The case of a two-to-one autoparametric resonance with the higher mode being excited by a primary resonance is considered. The results show that the response may exhibit jumps, saturation, and Hopf bifurcations.

20. **Nayfeh, A. H. and Balachandran, B., "Experimental Investigation of Resonantly Forced Oscillations of a Two-Degree-of-Freedom Structure," International Journal of Non-Linear Mechanics, November 1989.**

An experimental study of the response of a two-degree-of-freedom structure with quadratic nonlinearities and a two-to-one internal resonance to an external harmonic excitation is presented. When the excitation frequency was close to the lower natural frequency of the structure periodic, quasi-periodic, and chaotic responses were observed. Fourier spectra, time-dependent modal decompositions, and Poincáre maps were used to analyze the amplitude- and phase-modulated motions of the structure.

## *IV. Presentations*

1. Nayfeh, A. H., "Parametric Excitation of Two Internally Resonant Oscillators," Mathematics Seminar, VPI&SU, February 18, 1987.

2. Nayfeh, A. H., "Perturbation Methods in Nonlinear Dynamics," AFOSR/ARO Conference on Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanics, VPI&SU, March 23-25, 1987.

3. Nayfeh, A. H., Mook, D. T., and Nayfeh, J., "Some Aspects of Modal Interactions in the Response of Beams," AIAA/ASME/ASCE/AHS 28th Structures, Structural Dynamics and Materials Conference, Monterey, CA, April 6-8, l987.

4. Nayfeh, A H., "Can the Practicing Engineer Afford to Be Ignorant of Nonlinear Phenomena?," Electrical Engineering Seminar, VPI&SU, May 22, 1987.

5. Nayfeh, A. H. and Hamdan, A. M. A., "Modal Observability and Controllability Measures for First and Second Order Linear Systems and Model Reduction," presented at The VPI/AIAA Symposium on Dynamics and Control of Large Structures, VPI&SU, June 29-July 1, 1987.

6. Nayfeh, A. H., "Parametric Excitation of Two Internally Resonant Oscillators," presented at the International Conference on Nonlinear Oscillation, Budapest, Hungary, August 17-23, 1987.

7. Nayfeh, A. H., "Can the Practicing Engineer Afford to be Ignorant of Nonlinear Phenomena?," presented at the Second Technical Workshop on Dynamics and Aeroelastic Stability Modeling of Rotorcraft Systems, Boca Raton, FL, November 18-20, 1987.

8. Nayfeh, A. H. and Zavodney, L. D., "The Response of a Beam with Concentrated Mass to Parametric Excitation," presented at the Fifth Annual Review, Virginia Tech Center for Composite Material and Structures, VPI&SU, Blacksburg, VA, April 4-6, 1988.

9. Nayfeh, A. H. and Nayfeh, J., "Modal Interactions in the Response of a Beam to a Harmonic Excitation," presented at the Fifth Annual Review, Virginia Tech Center for Composite Material and Structures, VPI&SU, Blacksburg, VA, April 4-6, 1988.

10. Nayfeh, A. H., and Pai, F., "Complicated Responses of a Cantilevered Beam to a Harmonic Excitation at the Base," presented at the Fifth Annual Review, Virginia Tech Center for Composite Material and Structures, VPI&SU, Blacksburg, VA, April 4-6, 1988.

11. Nayfeh, A. H., "Analytical and Experimental Studies of Nonlinear Phenomena," presented at the 6th Annual Forum on Space Structures, Atlanta, GA, April 7-8, 1988.

12. Nayfeh, A. H. and Hamdan, A. M. A., "Interaction of Nonlinearities and Linear State Feedback," presented at the IEEE Conference and Exhibit, Knoxville, TN, April 10-13, 1988.

13. Nayfeh, A. H., "Modal Interactions in the Nonlinear Response of Structural Elements-Theory and Experiment," seminar given at Renssalear Polytechnic Institute, Troy, NY, May 4, 1988.

14. Nayfeh, A. H. and Asfar, K. R., "Nonstationary Parametric Oscillations," presented at the ASCE Engineering Mechanics Division Specialty Conference, VPI&SU, Blacksburg, VA, May 23-25, 1988.

15. Nayfeh, A. H. and Serhan, S., "The Response of the Duffing-Rayleigh Oscillator to a Random Excitation," presented at the ASCE Engineering Mechanics Division Specialty Conference, VPI&SU, Blacksburg, VA, May 23- 25, 1988.

16. Nayfeh, A. H. and Zavodney, L. D., "Parametric Resonances in Nonlinear Structural Elements: Theory and Experiment," presented at the ASCE Engineering Mechanics Division Specialty Conference, VPI&SU, Blacksburg, VA, May 23-25, 1988.

17. Nayfeh, A. H. and Sanchez, N. E., "Bifurcations in a Forced Softening Duffing Oscillator", Proceedings of the Second Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanisms Conference, VPI&SU, Blacksburg, VA, June 1-3, 1988.

18. Nayfeh, A. H. and Pai, P. F., "Nonlinear Nonplanar Parametric Responses of an Inextensional Beam," Proceedings of the Second Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanisms Conference, VPI&SU, Blacksburg, VA, June 1-3, 1988.

19. Nayfeh, A. H. and Neal, H. L., "Response of a Single-Degree-of-Freedom System to a Nonstationary Parametric Excitation - Theory and Experiment", Proceedings of the Second Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanisms Conference, VPI&SU, Blacksburg, VA, June 1-3, 1988.

20. Nayfeh, A. H. and Serhan, S. J., "A Generalized Method of Averaging for Determining the Response of Nonlinear Systems to Random Excitations", Proceedings of the Second Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanisms Conference, VPI&SU, Blacksburg, VA, June 1-3, 1988.

21. Nayfeh, A. H., Balachandran, B., Colbert, M. A., and Nayfeh, M. A., "Theoretical and Experimental Investigation of Complicated Responses of a Two-Degree-of-Freedom Structure", Proceedings of the Second

Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanisms Conference, VPI&SU, Blacksburg, VA, June 1-3, 1988.

22. Nayfeh, A. H., Nayfeh, J. F., and Mook, D. T., "Modal Interactions in the Response of Beams to a Harmonic Excitation", Proceedings of the Second Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanisms Conference, VPI&SU, Blacksburg, VA, June 1-3, 1988.

23. Nayfeh, A. H. and Zavodney, L. D. "The Nonlinear Response of a Slender Beam Carrying a Lumped Mass to a Principal Parametric Excitation," Proceedings of the Second Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanisms Conference, VPI&SU, Blacksburg, VA, June 1-3, 1988.

24. Nayfeh, A. H., "The Nonlinear Response of Dynamic Systems to Multifrequency Excitations," presented at the NSF Mechanics, Structures and Materials Engineering Division Dynamic Systems and Control Program Review, Washington, DC, June 8, 1988.

25. Nayfeh, A. H. and Zavodney, L. D., "Modal Interactions in the Nonlinear Response of Structural Elements- Theory and Experiment," presented at the Third International Conference on Recent Advances in Structural Dynamics, Southampton, England, July 18-22, 1988.

26. Nayfeh, A. H., "Numerical-Perturbation Methods in Mechanics," presented at the Symposium on Advances and Trends in Computational Structural Mechanics and Fluid Dynamics, Washington, DC, October 17-19, 1988.

27. Nayfeh, A. H., "Modal Interactions in the Nonlinear Response of Structural Elements- Theory and Experiment," seminar given at the University of Michigan, Ann Arbor, MI, November 16, 1988.

28. Nayfeh, A. H., "Modal Interactions in Systems with Quadratic Nonlinearities", presented at the Pan-American Congress of Applied Mechanics, Sponsored by AAM, Rio De Janeiro, Brazil, January 3-6, 1989.

29. Nayfeh, A. H. and Balachandran, B., "Nonlinear Modal Interactions in a Composite Structure," presented at the Sixth Annual Review for the Center for Composite Materials and Structures, Blacksburg, VA, April 9-11, 1989.

30. Nayfeh, A. H., Nayfeh, J. F., and Mook, D. T., "Nonlinear Dynamic Response of Laminated Composite Plate Strips in Cylindrical Bending," presented at the Sixth Annual Review for the Center for Composite Materials and Structures, Blacksburg, VA, April 9-11, 1989.

31. Nayfeh, A. H. and Pai, P. F., "Nonlinear Nonplanar Parametric Responses of an Inextensional Beam," presented at the 7th VPI&SU/AIAA Symposium on Dynamics and Control of Large Structures, Blacksburg, VA, May 8-10, 1989.

32. Nayfeh, A. H. and Balachandran, B., "Modal Interactions in Resonantly Forced Structures," presented at the 7th VPI&SU/AIAA Symposium on Dynamics and Control of Large Structures, Blacksburg, VA, May 8-10, 1989.

33. Nayfeh, A. H., El-Zein, M. S., and Nayfeh, J. F., "Nonlinear Oscillations of Composite Plates Using Perturbation Techniques," presented at the 4th Technical Conference on Composite Materials, Virginia Polytechnic Institute and State University, Blacksburg, VA, October 3-6, 1989.

34. Nayfeh, A. H., "Modal Interactions in the Nonlinear Response of Dynamic and Structural Systems," presented at Purdue University as a Graduate Colloquium Speaker, West Lafayette, IN, October 19, 1989.

35. Nayfeh, A. H., Nayfeh, J. F., and Mook, D. T., "Nonlinear Vibration of Laminated Composite Plate Strips in Cylindrical Bending," presented at the 60th Shock and Vibration Symposium, Virginia Beach, VA, November 14-16, 1989.

36. Nayfeh, A. H., "Recent Advances in Nonlinear Oscillations," presented at Georgia Institute of Technology, Atlanta, GA, October 31, 1989.

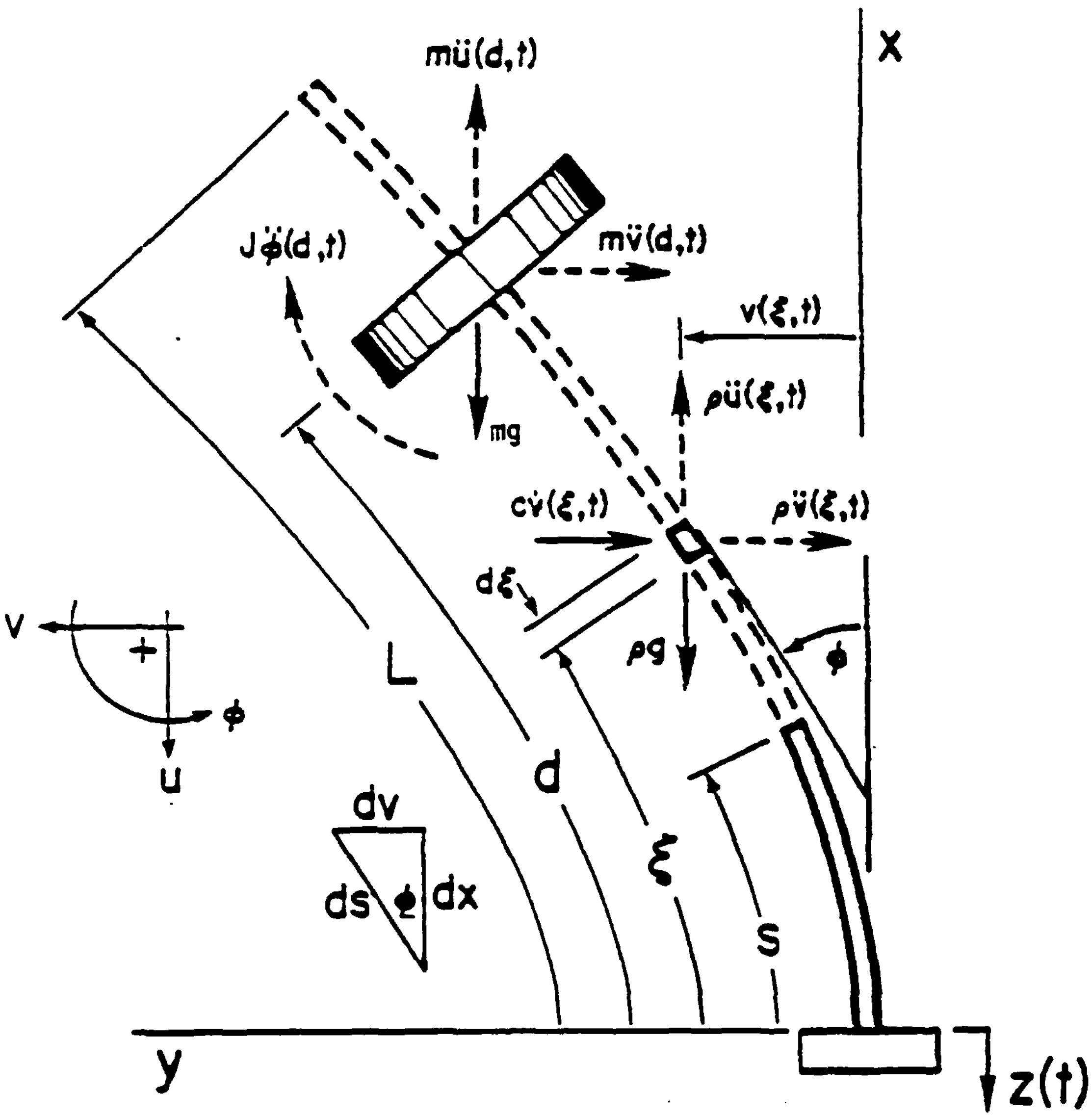


**Figure 1.** Cantilevered beam with a concentrated mass subjected to vertical base motion.

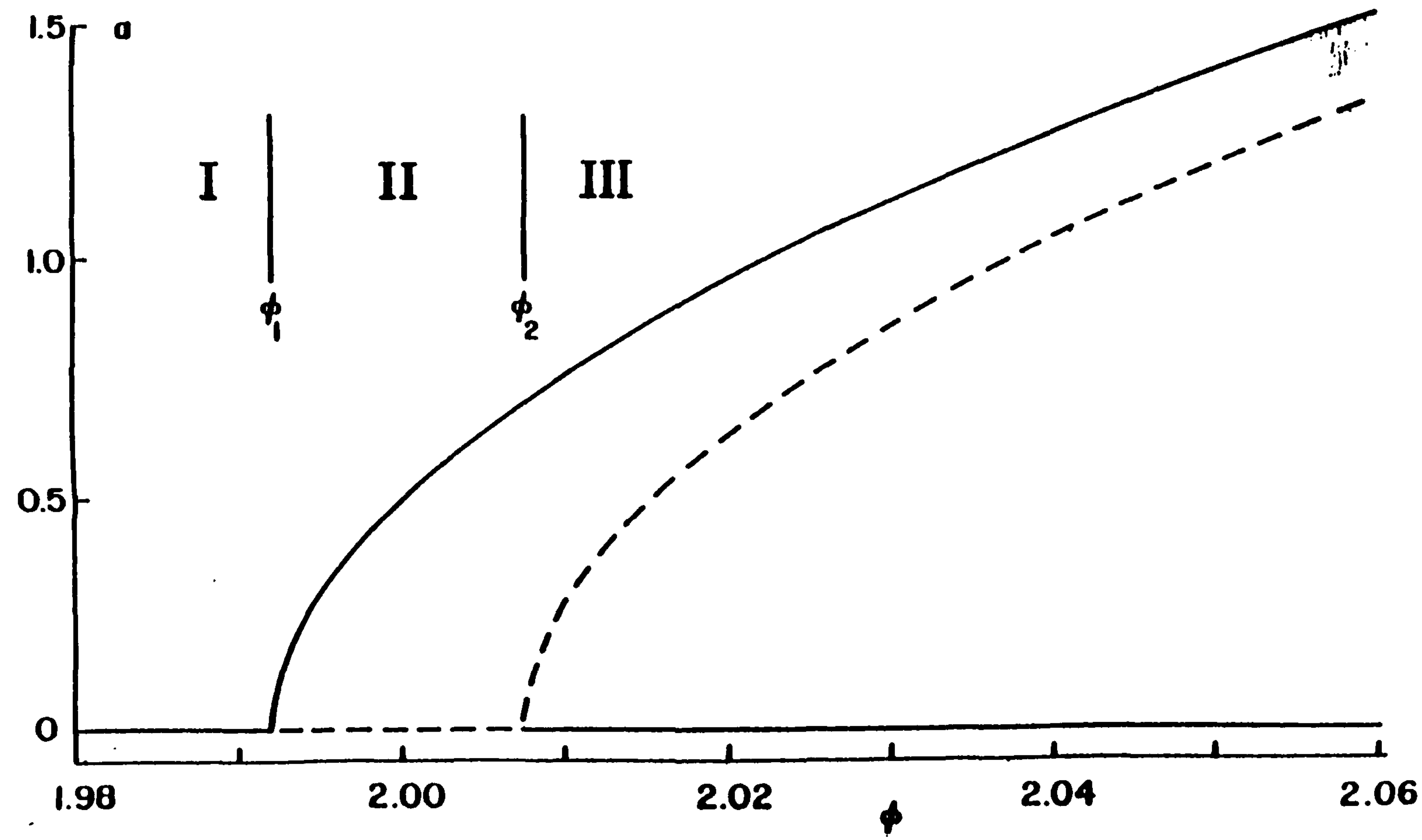


Figure 2. Theoretically determined frequency-response curve.

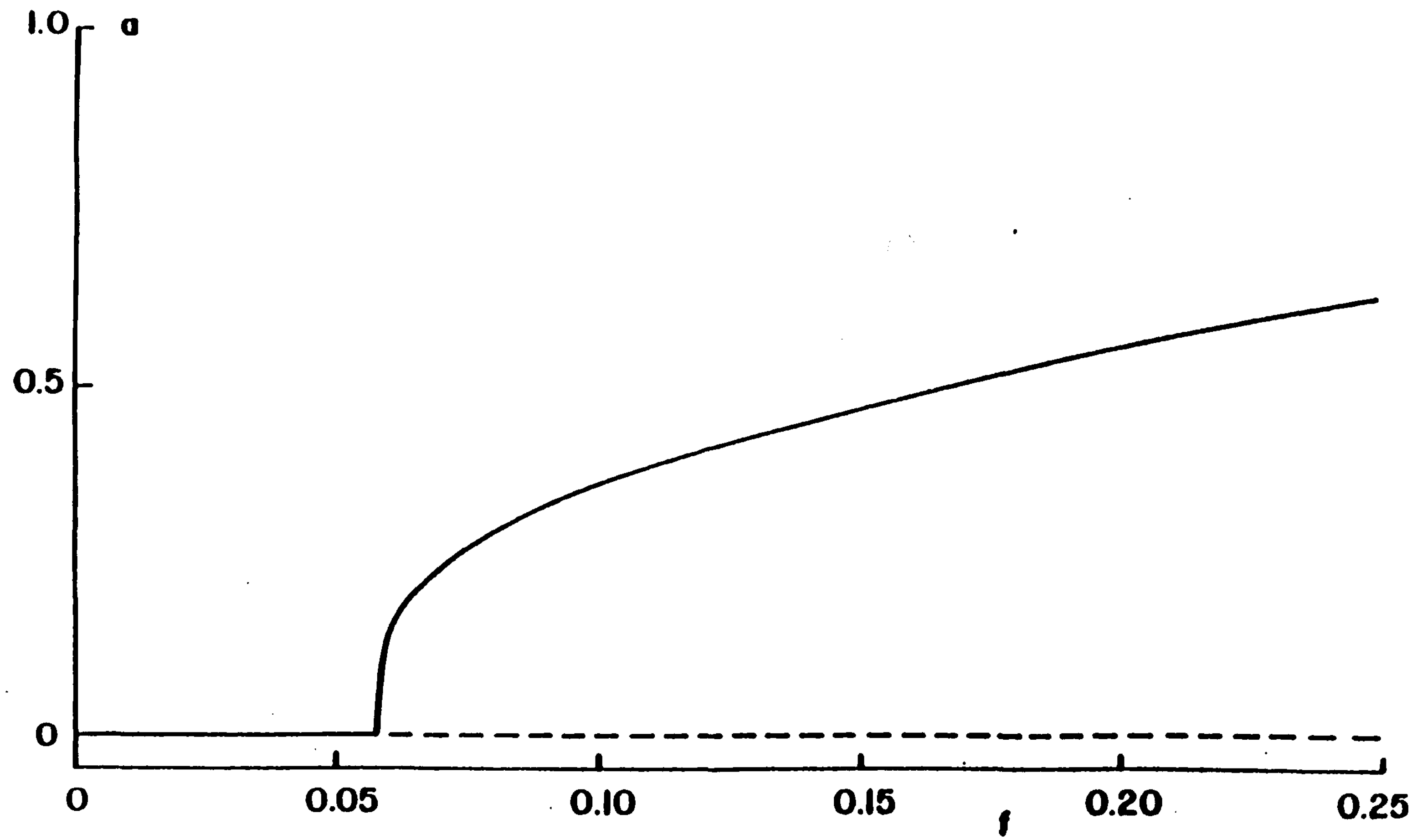


**Figure 3.** Theoretically determined variation of the steady-state amplitude a with the amplitude of excitation f in region II of Figure 2: $\phi = 2.000$ .

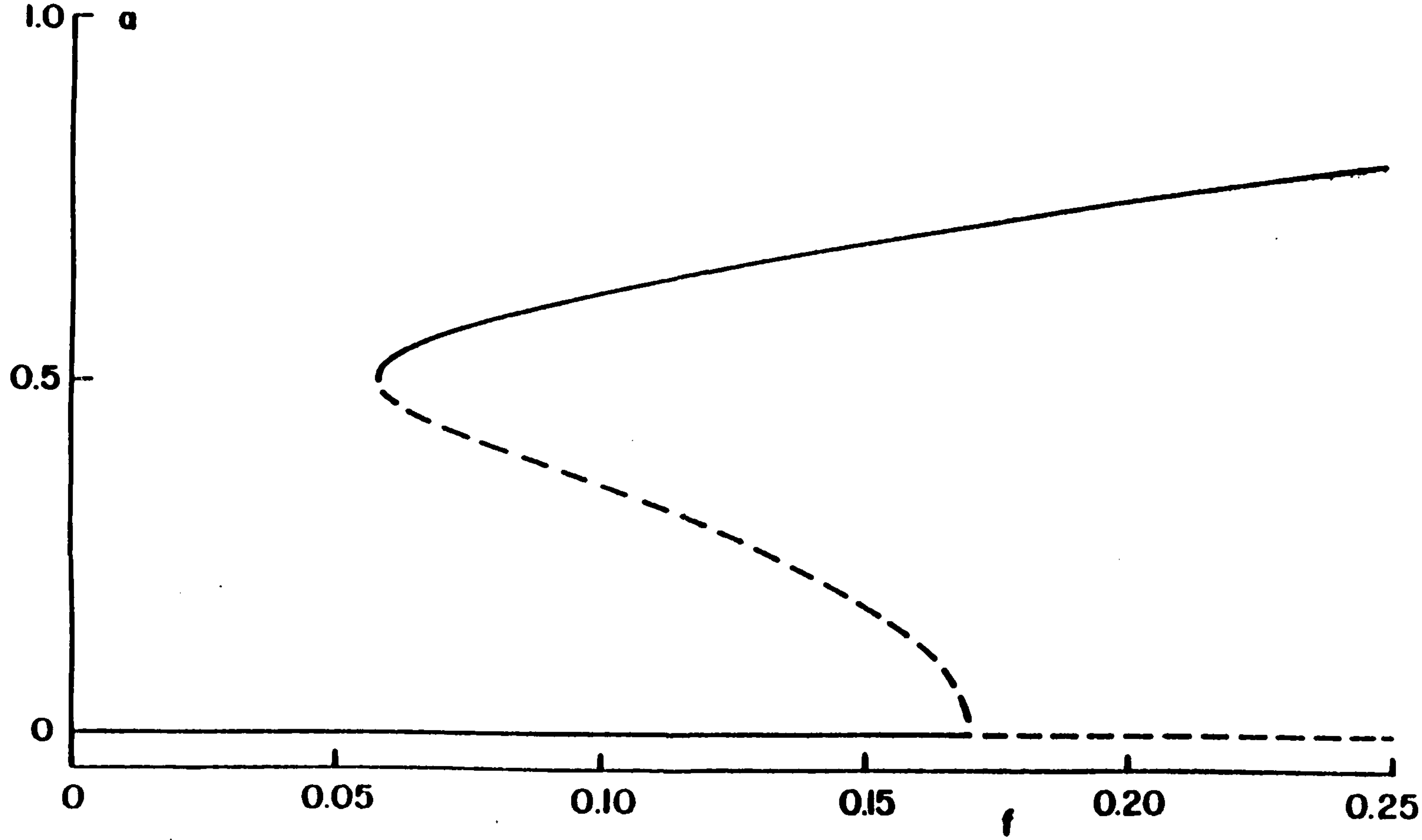


**Figure 4.** Theoretically determined variation of the steady-state amplitude a with the amplitude of excitation f in region III of Figure 2.

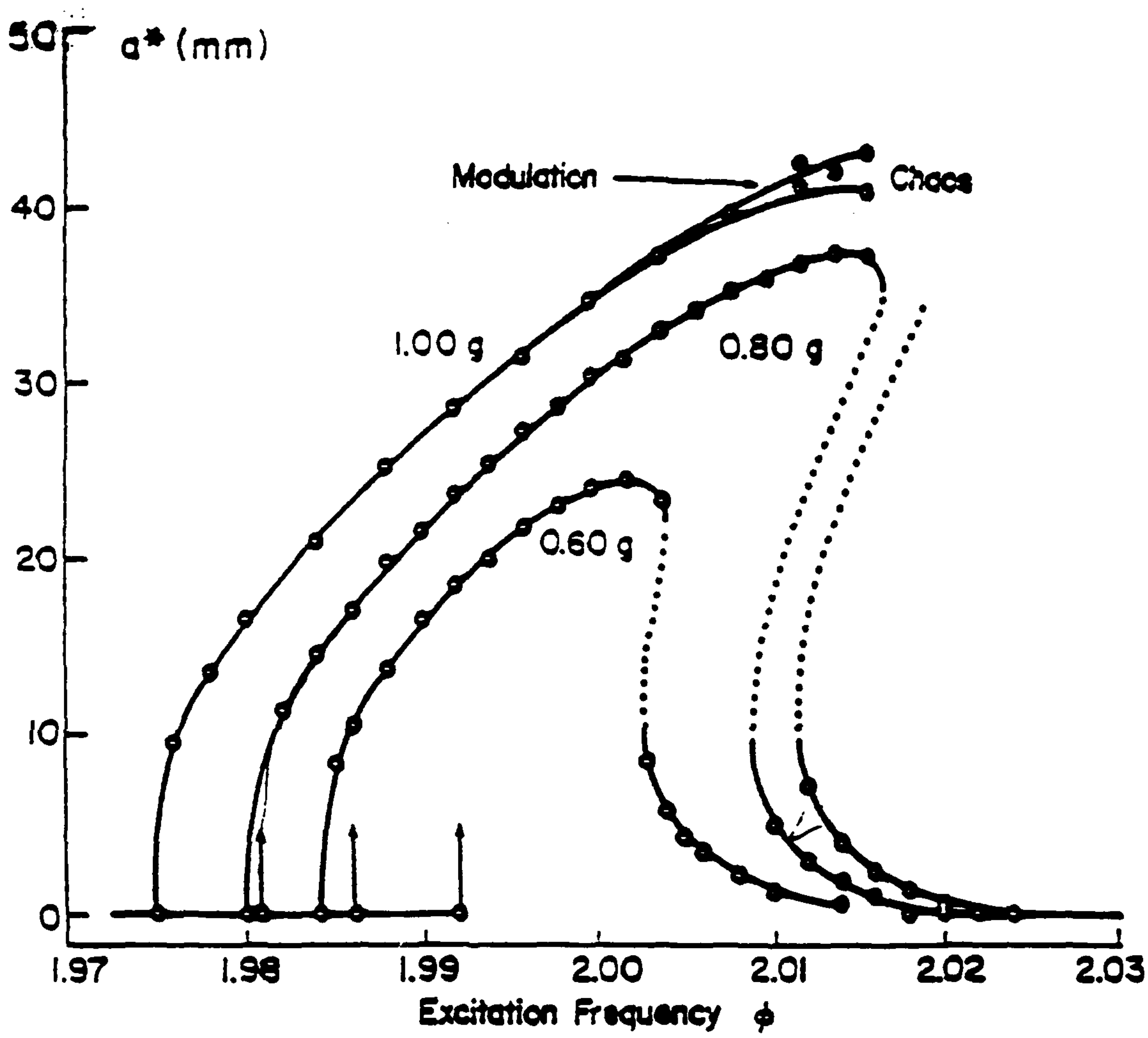


**Figure 5.** Frequency-response curves for three levels of excitation amplitude f of the composite beam. Note that chaotic behavior occurs at the largest amplitude of excitation.

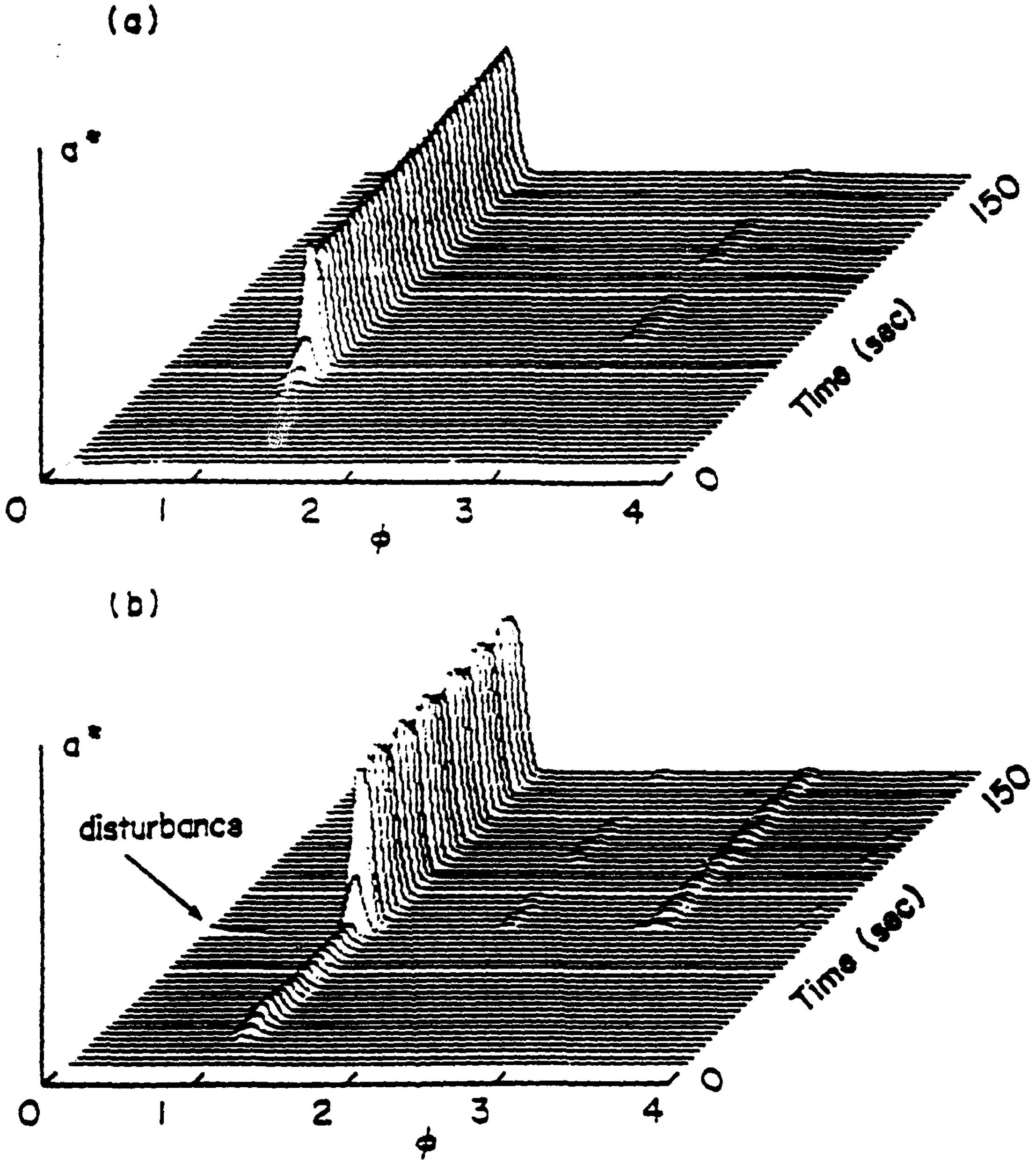


**Figure 6.** Spectral time history of the composite beam to a principal parametric excitation: (a) $\phi = 2.000$, (b) $\phi = 2.013$. Both responses start with the trivial solution. When $\phi = 2.013$ two solutions are possible.

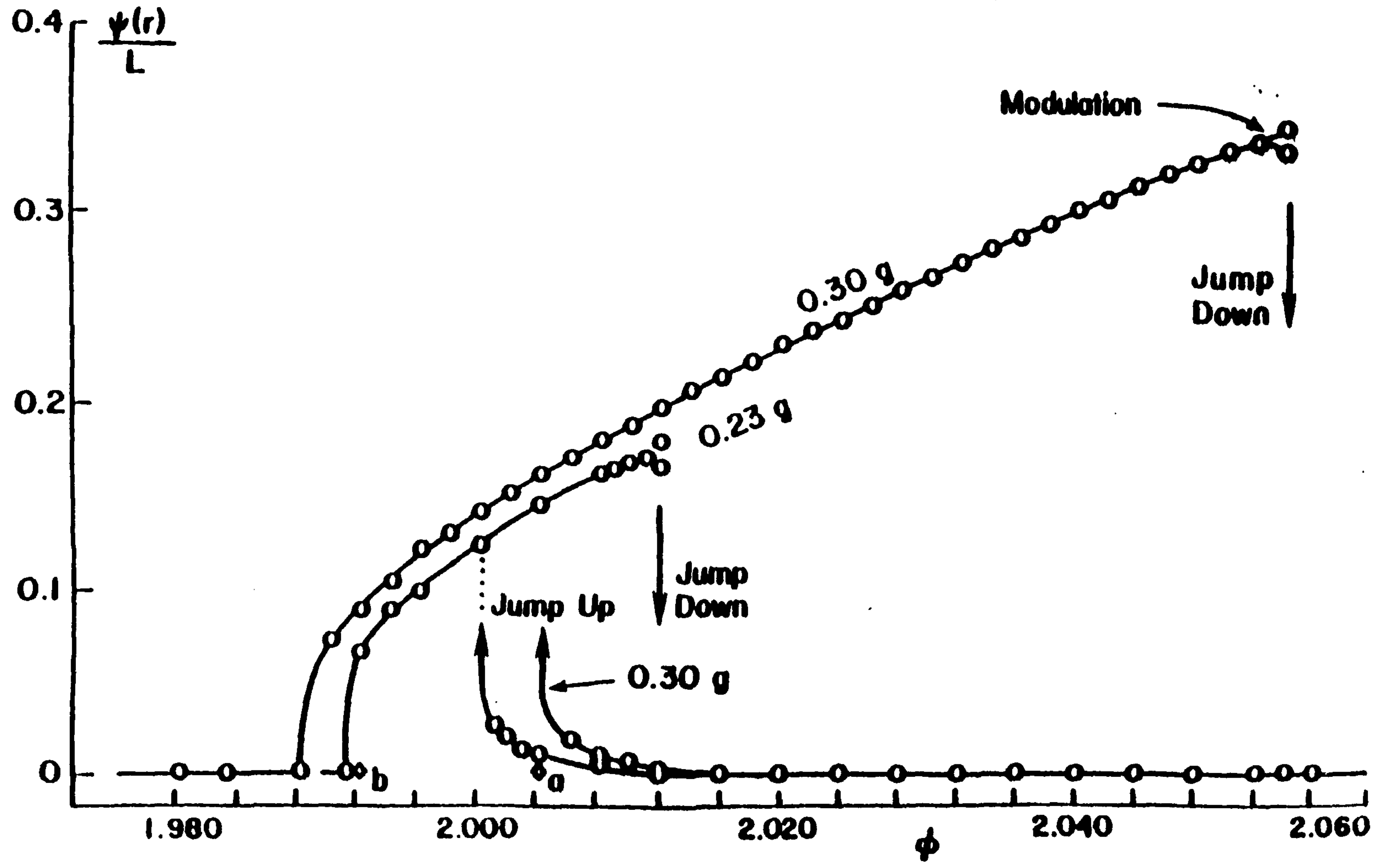


**Figure 7.** Frequency-response curves for the metallic beam shown in Figure 1 for two acceleration levels.

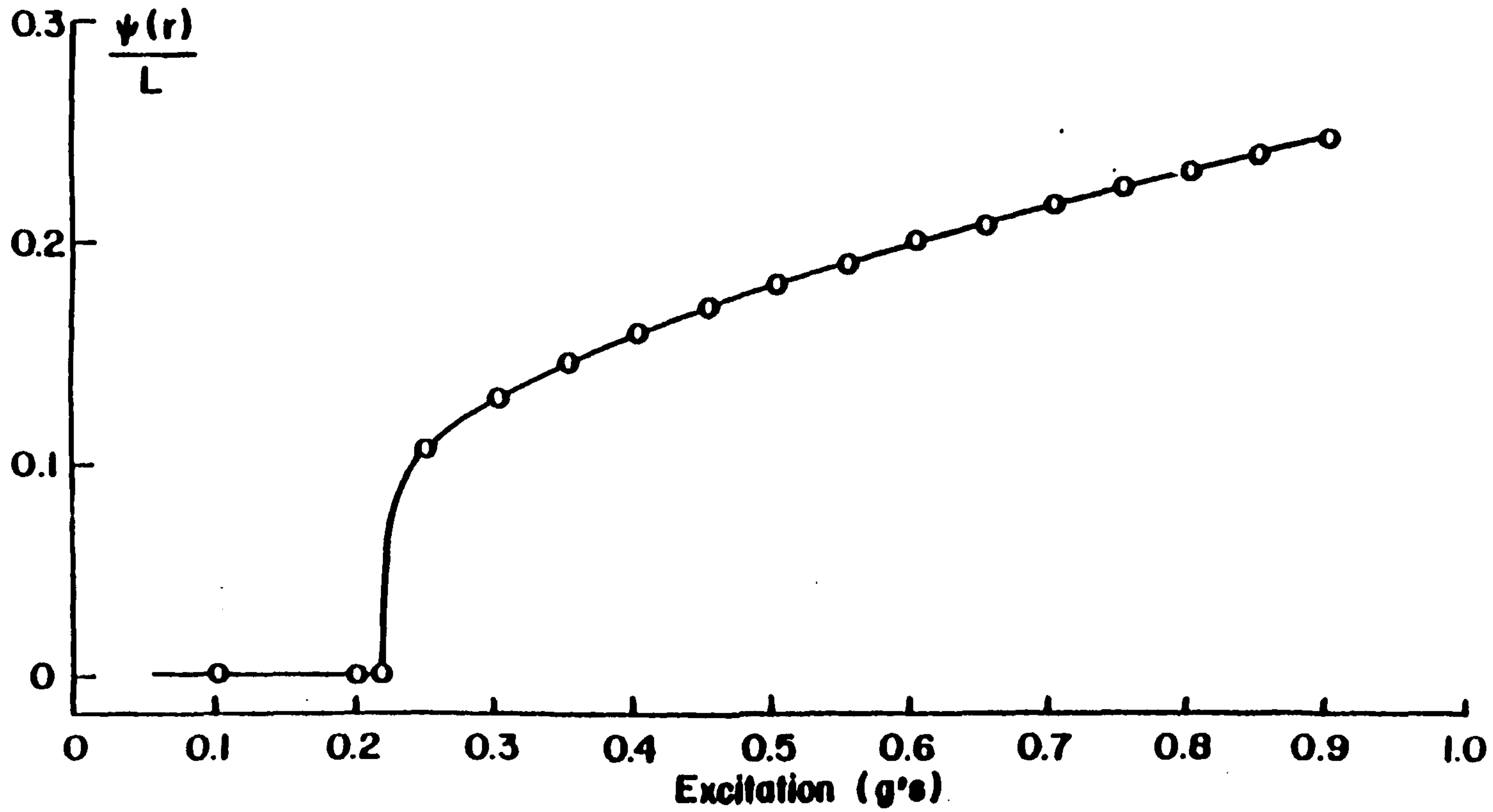


**Figure 8.** Variation of the amplitude a* with the excitation amplitude f of the beam shown in Figure 1 for $\phi = 2.000$.

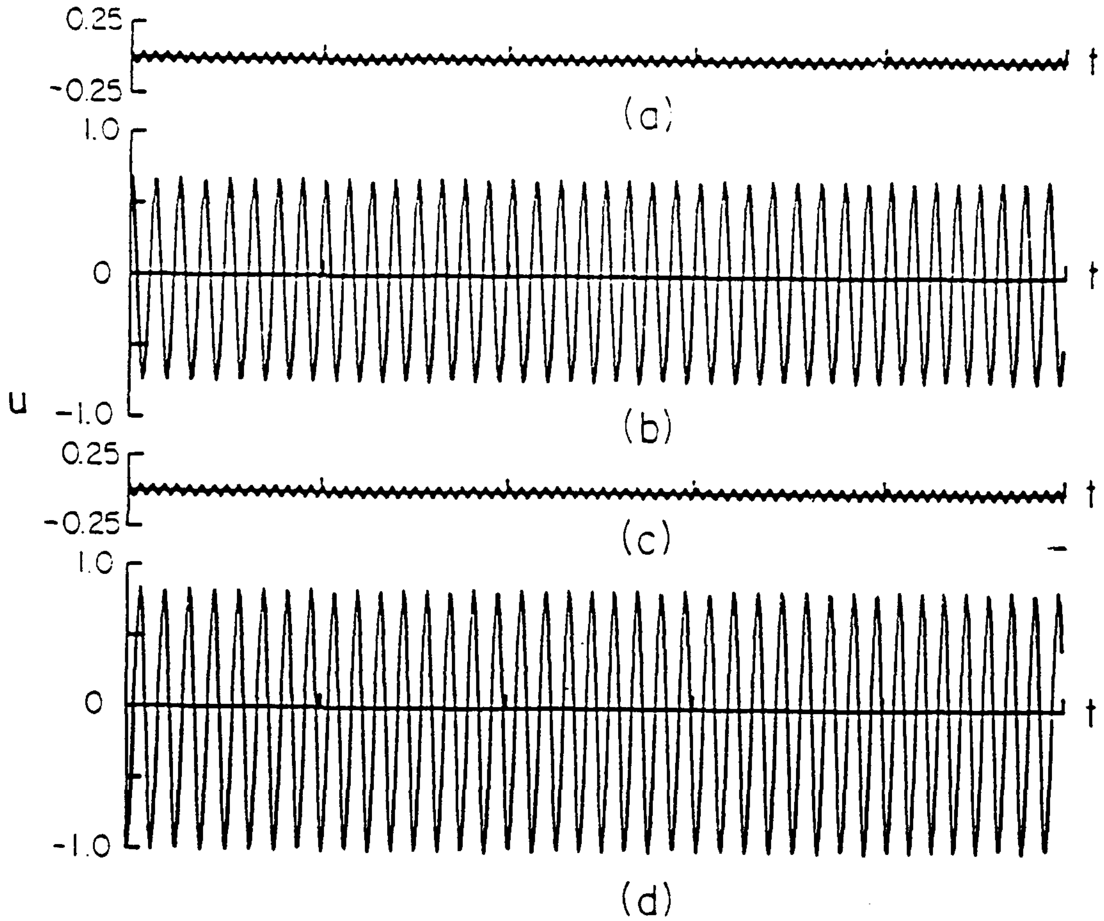


**Figure 9.** Response curves demonstrating the quenching and enhancement of the principal parametric response by the addition of a subharmonic excitation of order one-half; (a) response to subharmonic excitation only, (b) response to parametric excitation only, (c) response to both excitation with $\tau = 0$, and (d) response to both excitation with $\tau = \pi$.

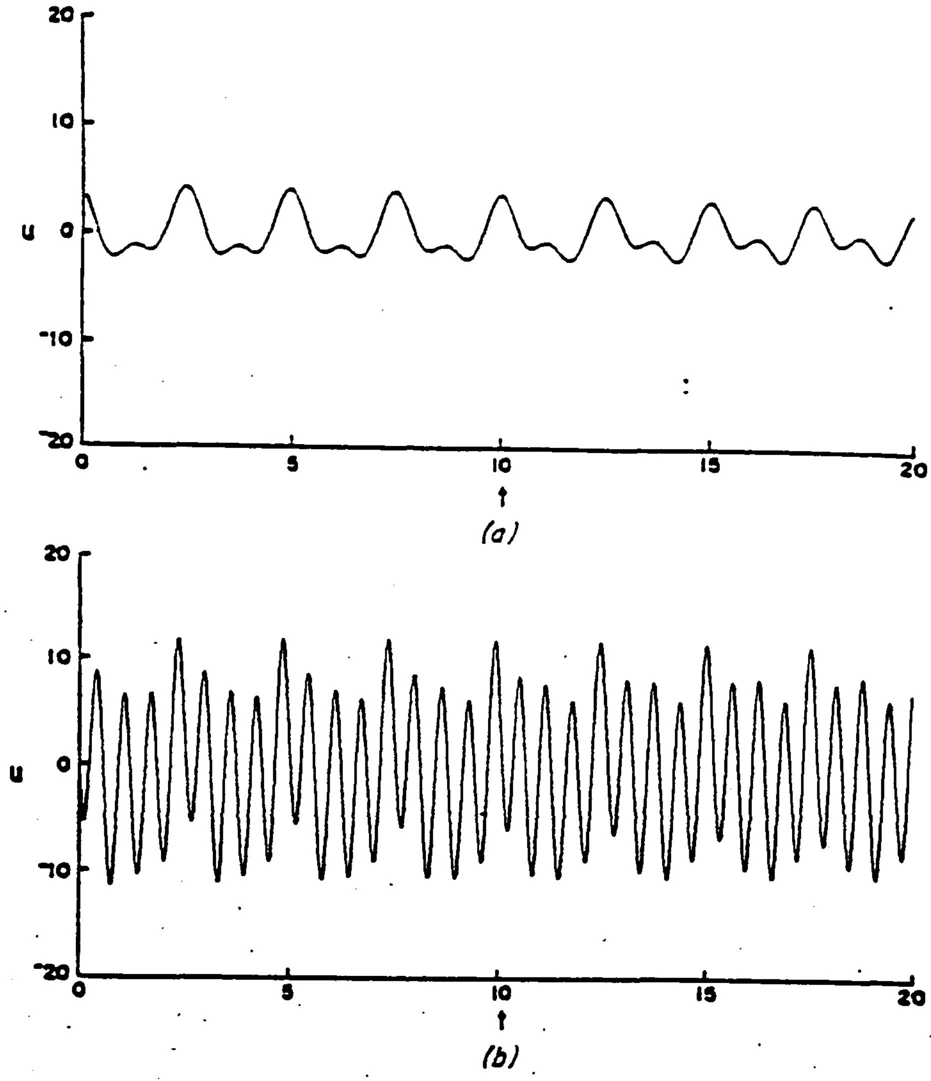


**Figure 10.** Response of the Duffing equation to a three-frequency excitation: (a) linear case; (b) nonlinear case.

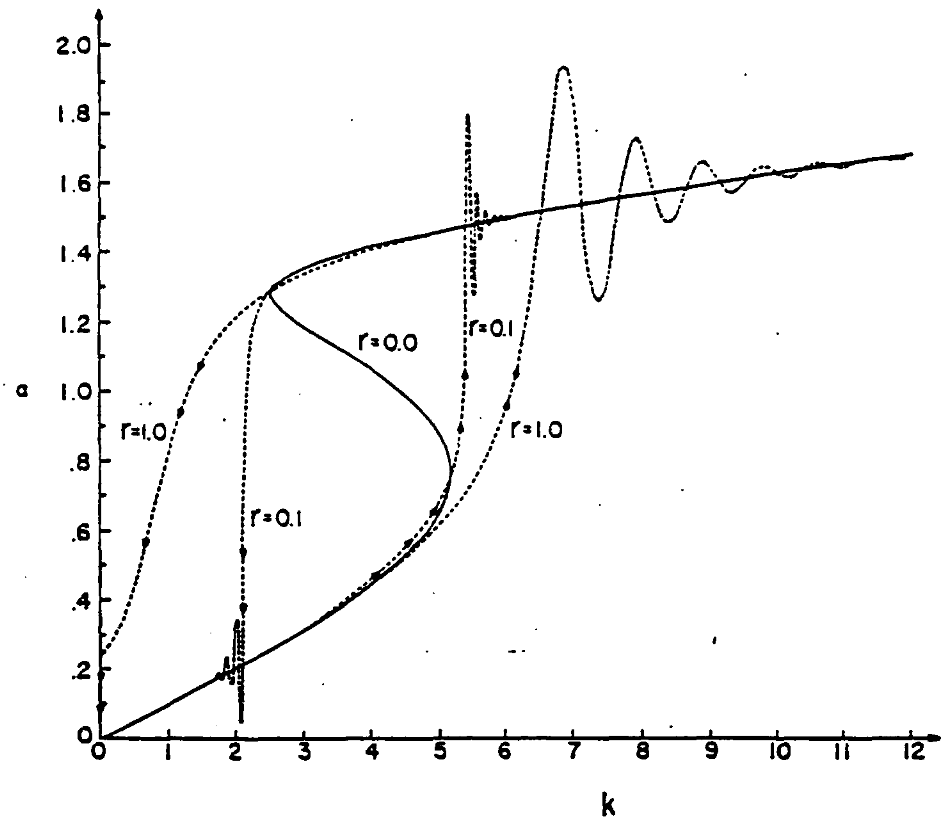


**Figure 11.** Comparison of nonstationary and stationary response curves: (__________) stationary results: (- - - - -) nonstationary results for several rates of changing k.

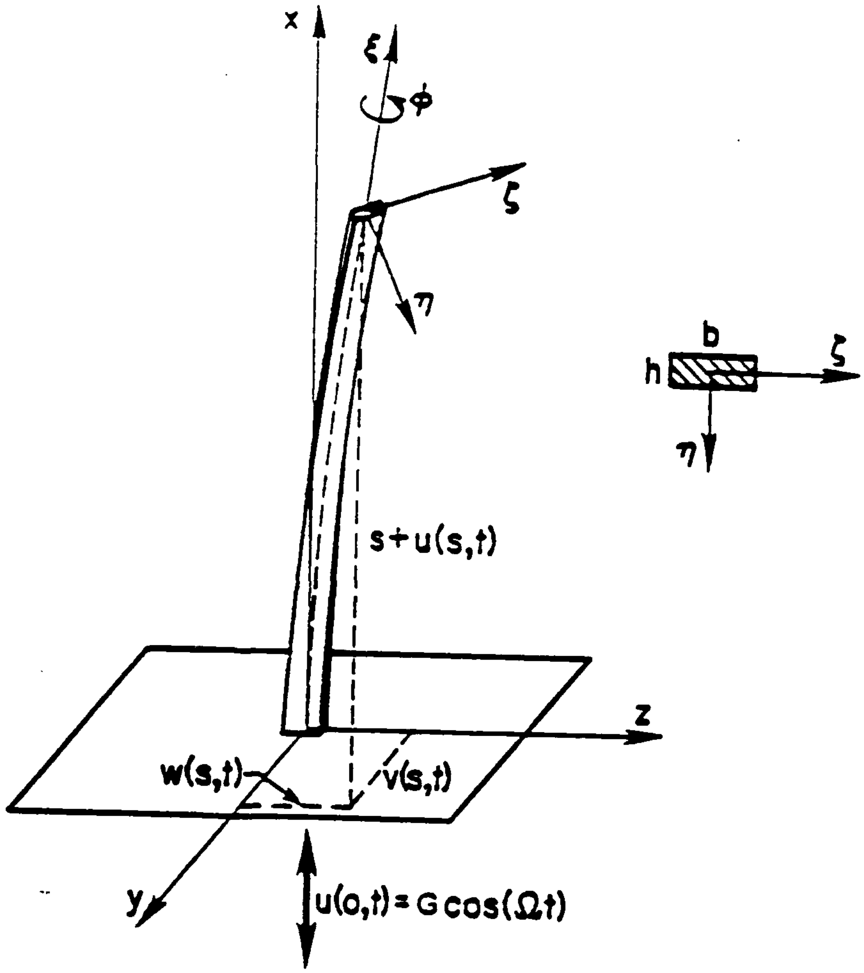


**Figure 12.** Coordinate systems: x-y-z = the inertial reference frame; $\xi - \nu - \zeta$ = the principal axes of the beam's cross section at position s, which is fixed on the cross section. .

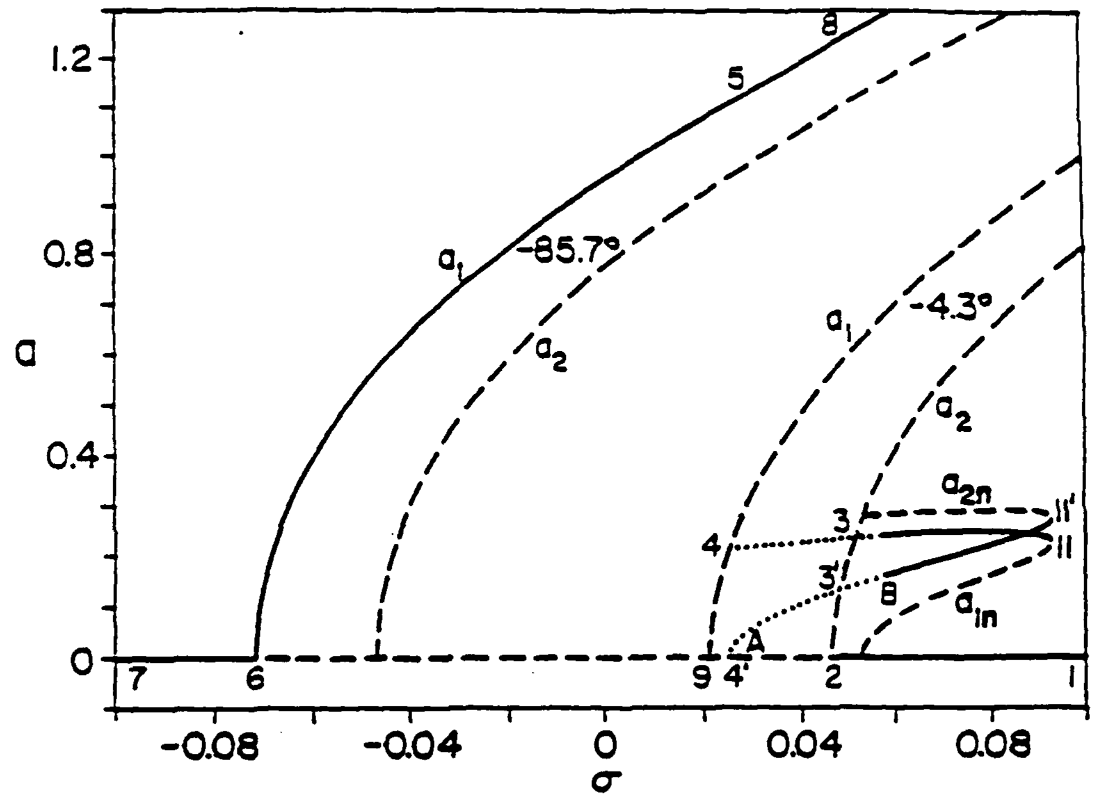


**Figure 13.** Response curves of the first mode for a beam with an aspect ratio $\hat{b}/h \neq 1.0$,: mode (1,1) $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.05$, $\mu = 0.05$, $b = 0.03$, $\beta_y = 0.7692$; a sub 1 , a sub 2 = planar response amplitudes; $a_{1n}$, $a_{2n}$ = nonplanar response amplitudes; (____) stable, ( - - - ) unstable with at least one eigenvalue being positive, (. . . ) unstable with the real part of a complex pair of eigenvalues being positive.

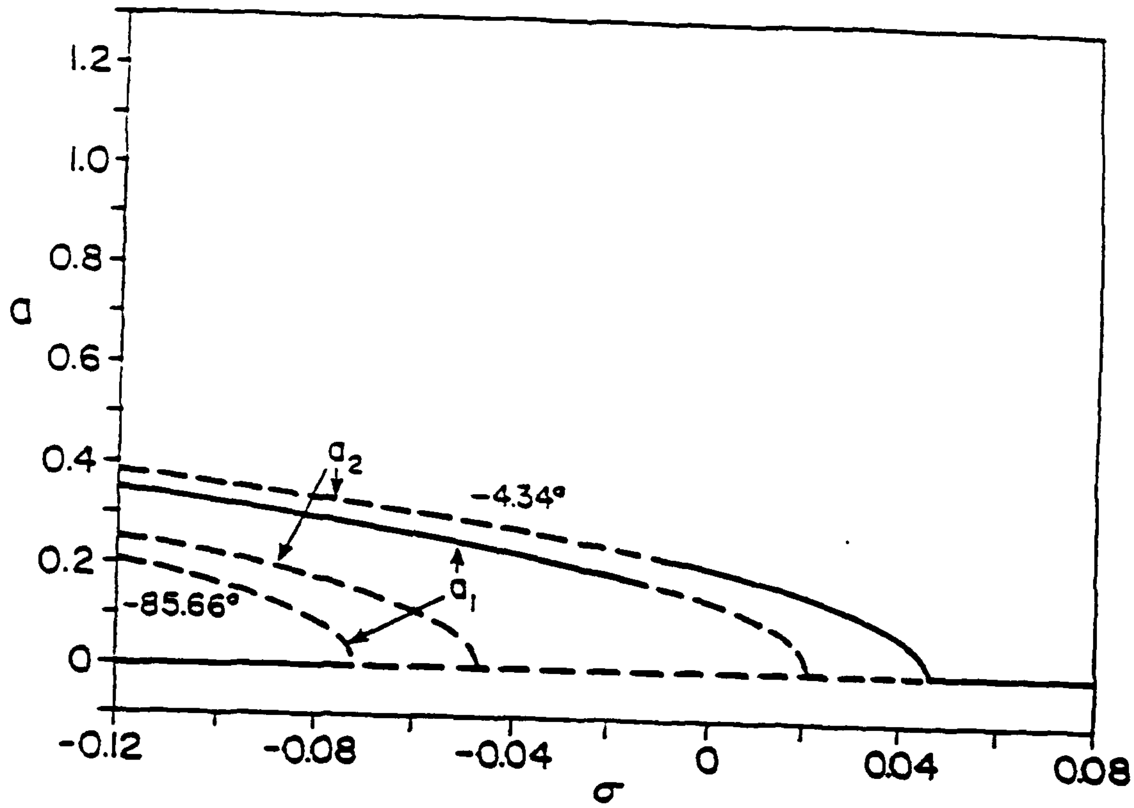


**Figure 14.** Response curves of the first mode in the absence of the nonlinear geometric terms, for a beam with an aspect ratio $\hat{b}/h \neq 1.0$,: mode (1,1) $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.05$, $\mu = 0.05$, $b = 0.03$, $\beta_y = 0.7692$; a sub 1 , a sub 2 = planar response amplitudes; (____) stable, ( - - - ) unstable with at least one eigenvalue being positive.

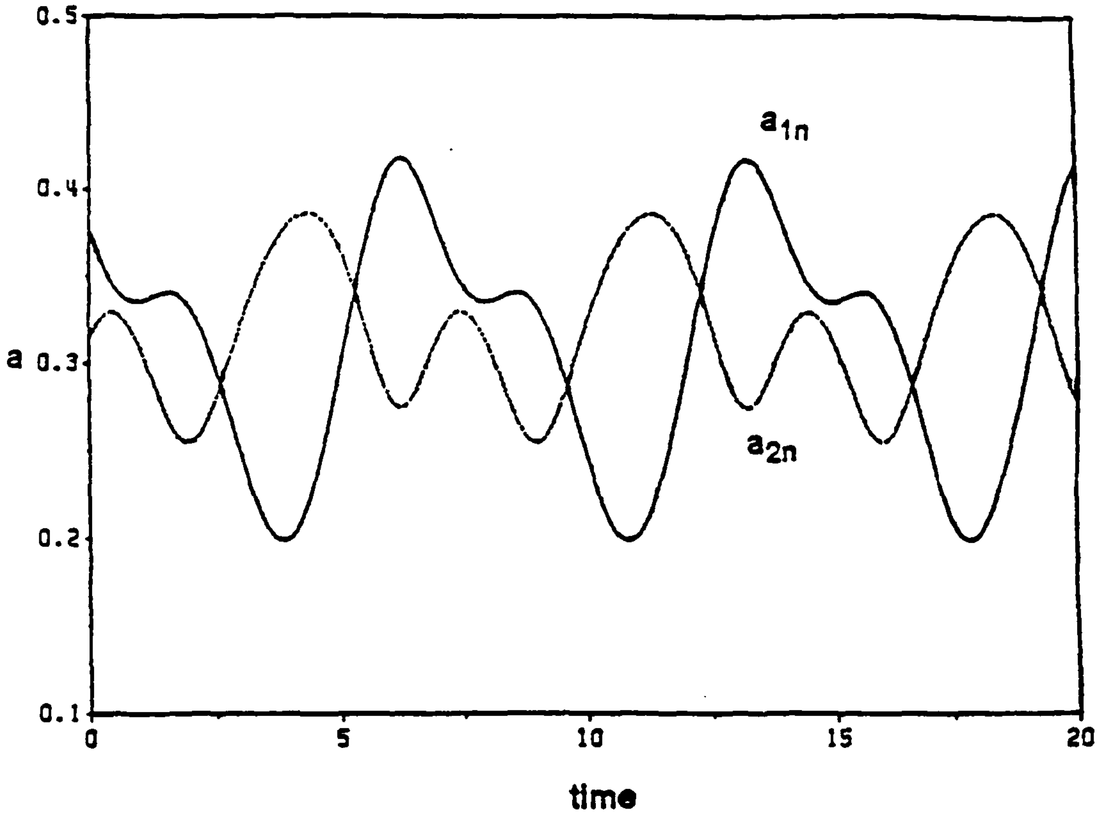


**Figure 15.** The long-time history of the amplitudes for the case of an amplitude- and phase-modulated motion: $\hat{b}/h \neq 1.0$,: mode (1,1) $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.05$, $\mu = 0.05$, $b = 0.03$, $\beta_y = 0.7692$, $\sigma = -0.0353$.

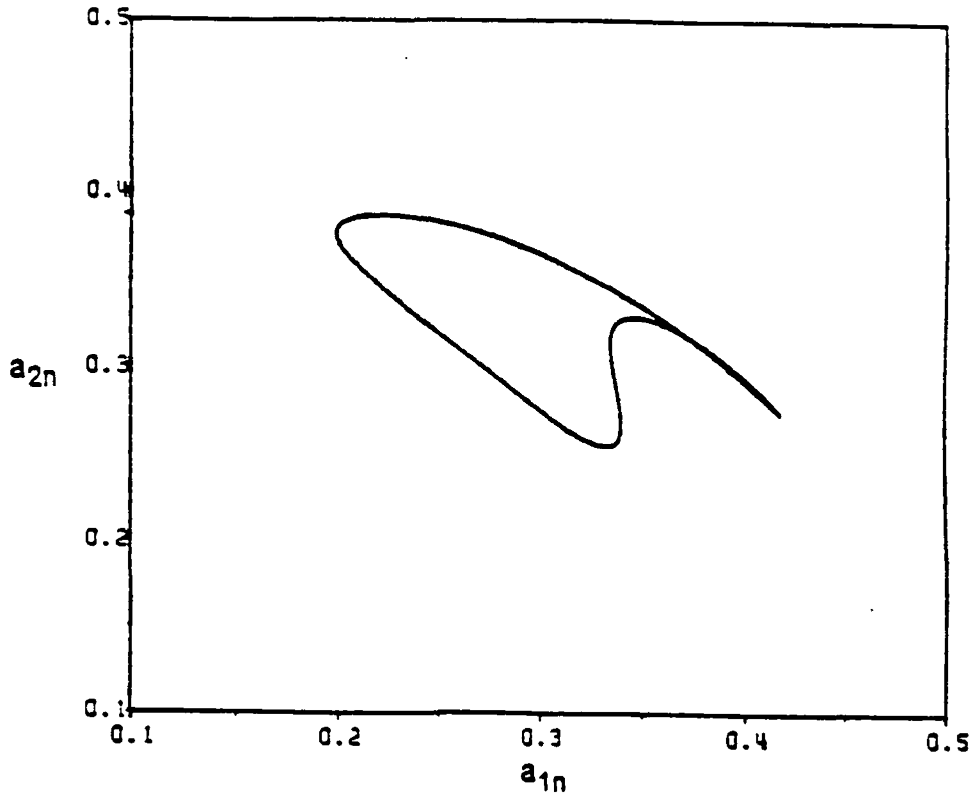


**Figure 16.** A projection of the trajectory onto the $a_1 - a_2$ plane: $\hat{b}/h \neq 1.0$,: mode (1,1) $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.05$, $\mu = 0.05$, $b = 0.03$, $\beta_y = 0.7692$, $\sigma = -0.0353$.

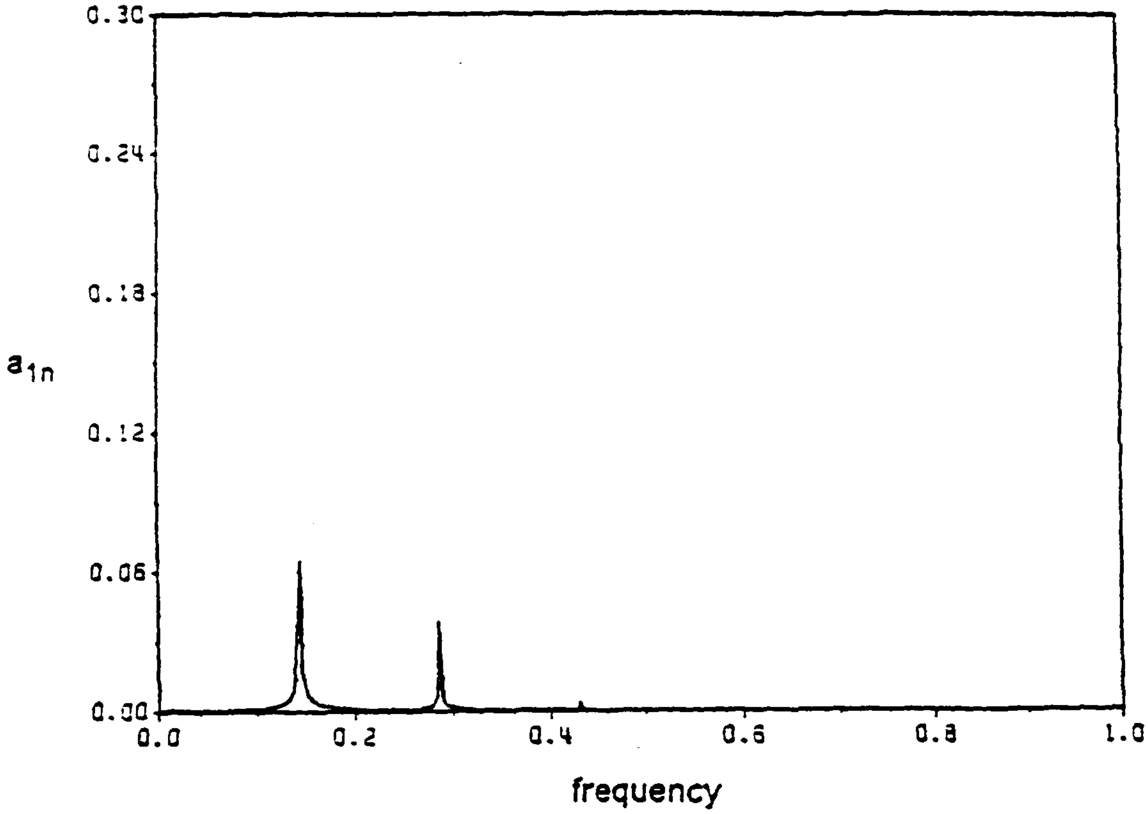


**Figure 17.** The Fourier harmonic analysis of $a_{1n}$: $\hat{b}/h \neq 1.0$,: mode (1,1) $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.05$, $\mu = 0.05$, $b = 0.03$, $\beta_y = 0.7692$, $\sigma = -0.0353$.

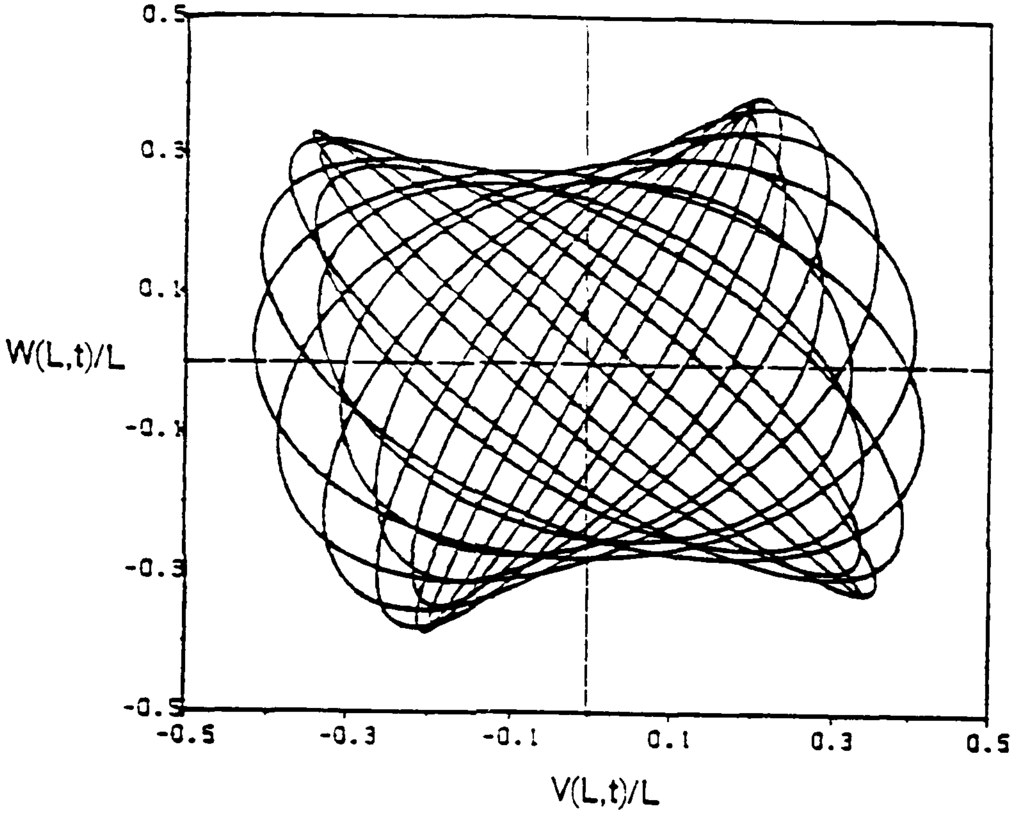


**Figure 18.** The path of the tip-end of the beam for the case of an amplitude- and phase-modulated motion: $\hat{b}/h \neq 1.0$,: mode (1,1) $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.05$, $\mu = 0.05$, $b = 0.03$, $\beta_y = 0.7692$, $\sigma = -0.0353$, $\varepsilon = 0.5$.

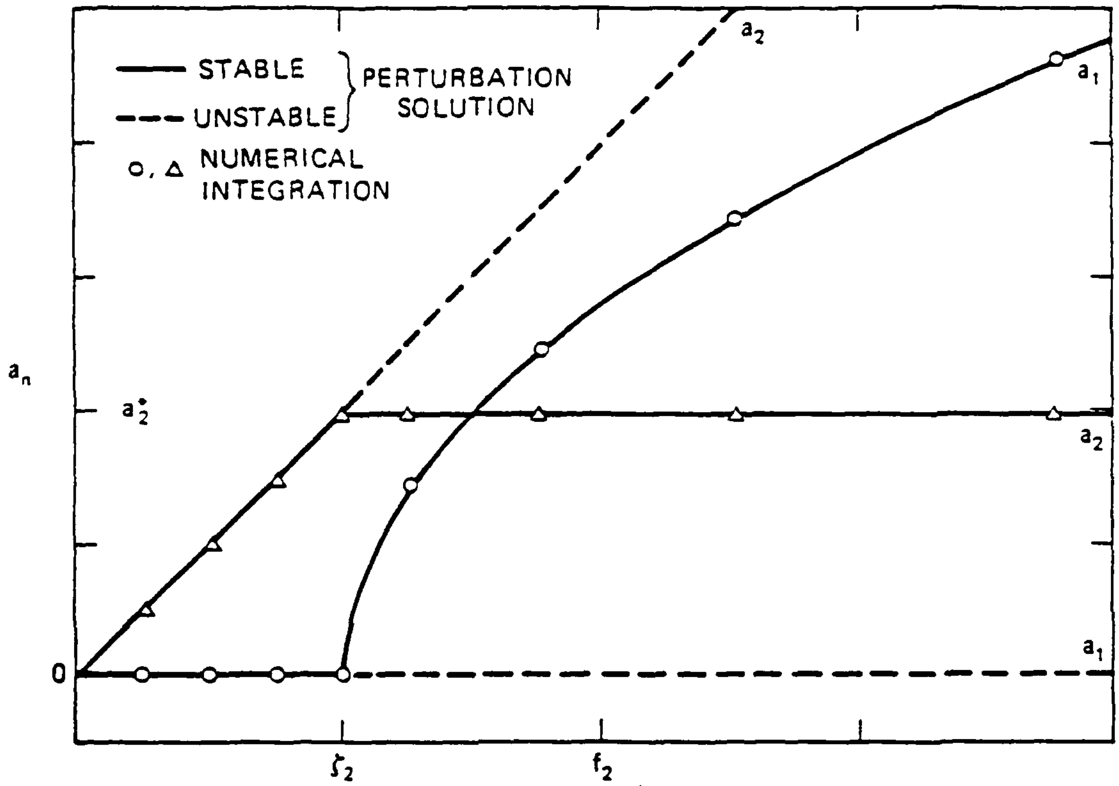


**Figure 19.** Variations of the modal amplitudes $a_1$ and $a_2$ with the amplitude $f_2$ of a primary excitation of the second mode when the second natural frequency is twice the first natural frequency. It demonstrates the instability of the linear solution and the occurrence of the saturation phenomenon.

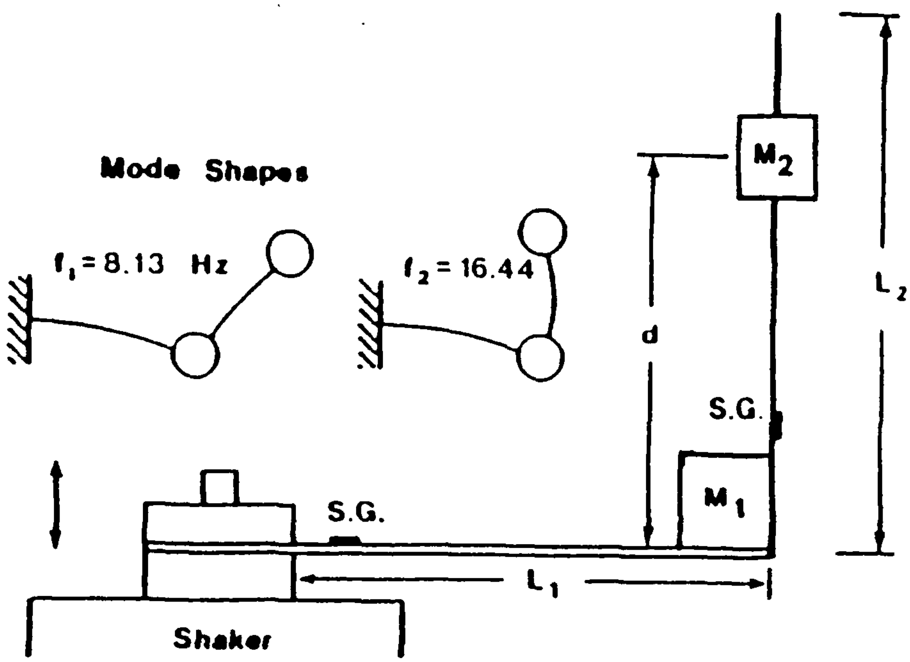


**Figure 20.** Two-degree-of-freedom model tuned for a 2:1 internal resonance and accompanying linear mode shapes. Beam 1: 1.676mm x 12.827mm x 154.51mm, $\rho_1$ = 0.162g/mm, $m_1$ = 33.1g; Beam 2: 0.559mm x 12.802mm x 152.40mm, $\rho_2$ = 0.0498g/mm, $m_2 = 40.0g$ ; d = 90.525mm.

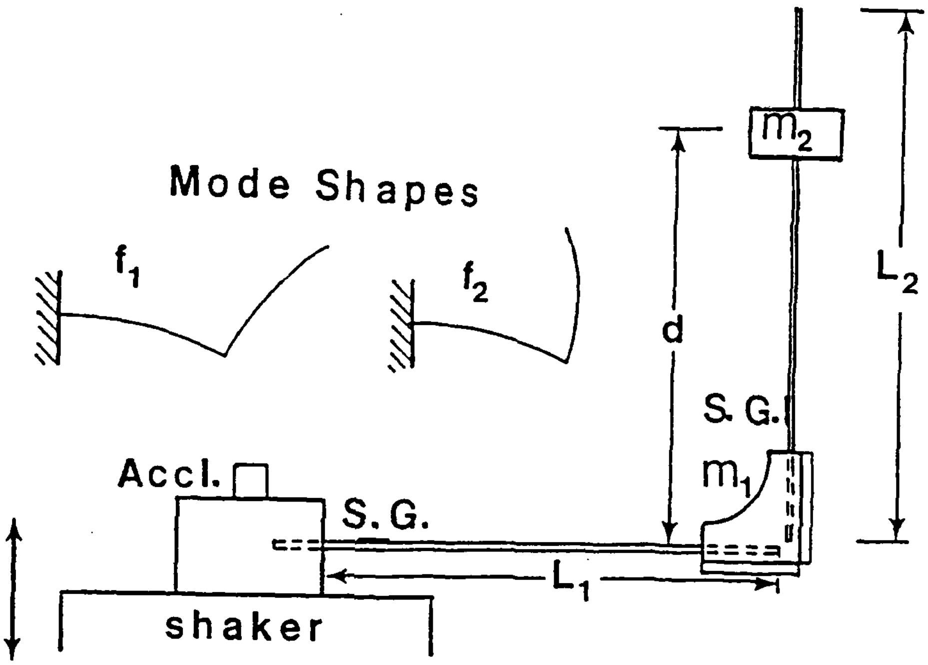


**Figure 21.** Composite structure tuned for internal resonance and accompanying flexural mode shapes. Beam 1: 2.159mm x 12.954mm x 193.04mm, $\rho_1$ = 0.0586g/mm, $m_1$ = 11.7g; Beam 2: 2.159mm x 12.945mm x 203.20mm, $\rho_2$ = 0.0531g/mm, $m_2 = 14.7g$ ; d = 152.97mm.

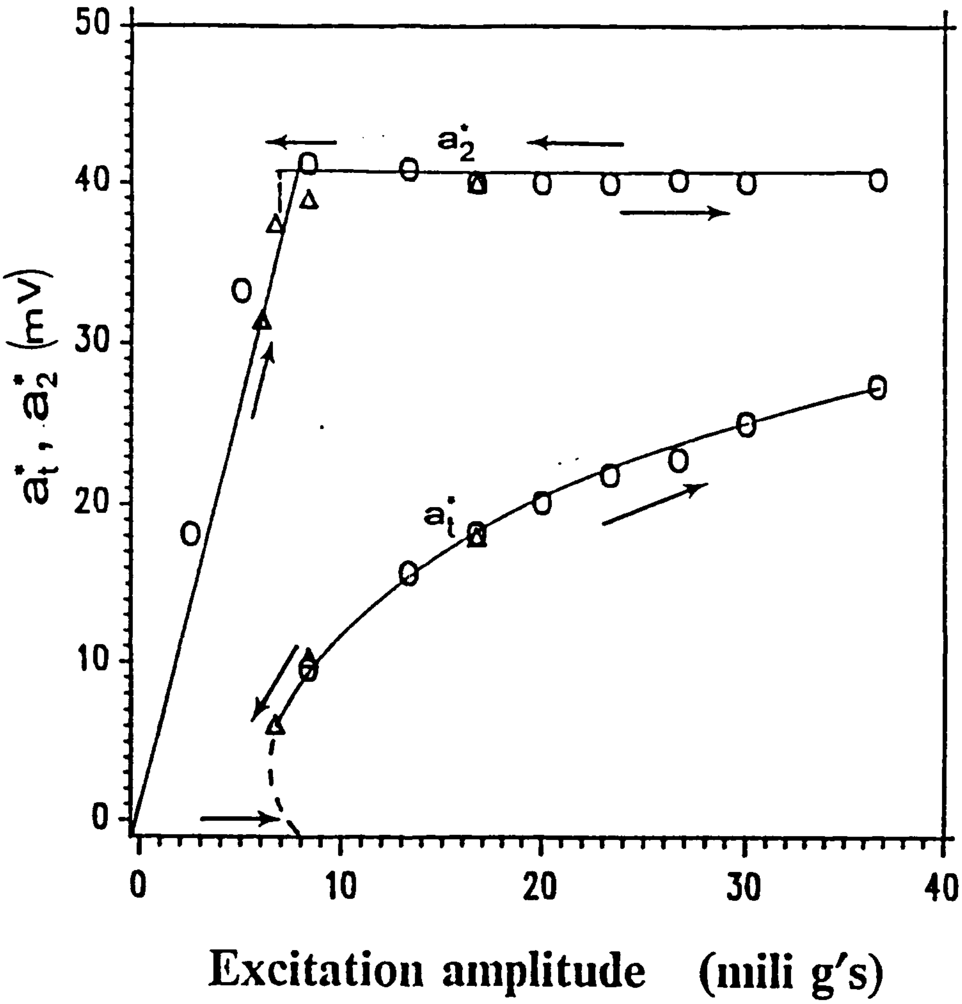


**Figure 22.** Amplitude-response curves when the excitation frequency is held constant at 16.80 Hz.

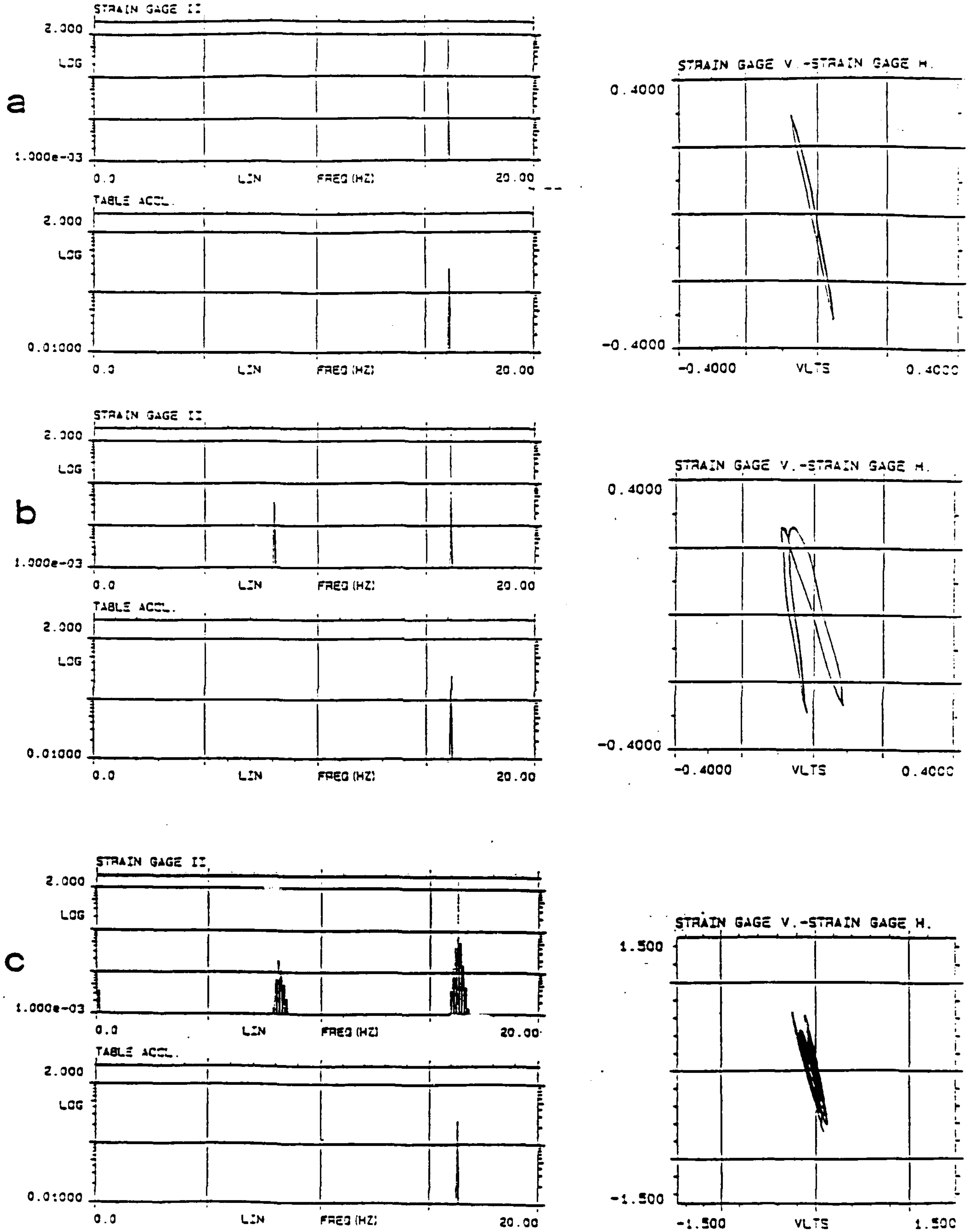


**Figure 23.** Spectra of the response, excitation, and cross-plot of two strain-gage signals: (a) f = 16.10 Hz, 9b) f = 16.14, and (c) f = 16.30 Hz.

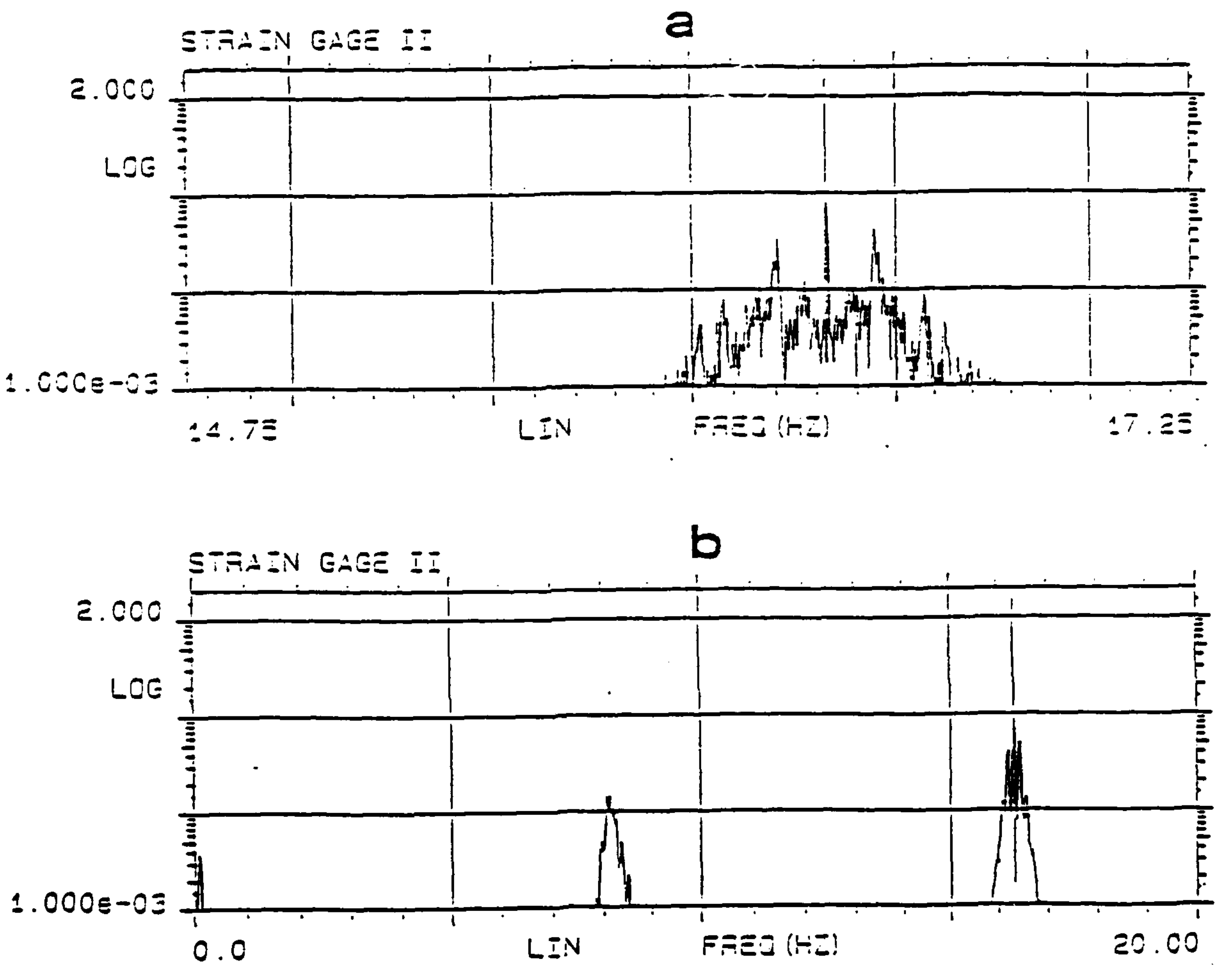


**Figure 24.** Spectra of a chaotic response: (a) zoom span around f = 16.325 Hz and (b) base band.

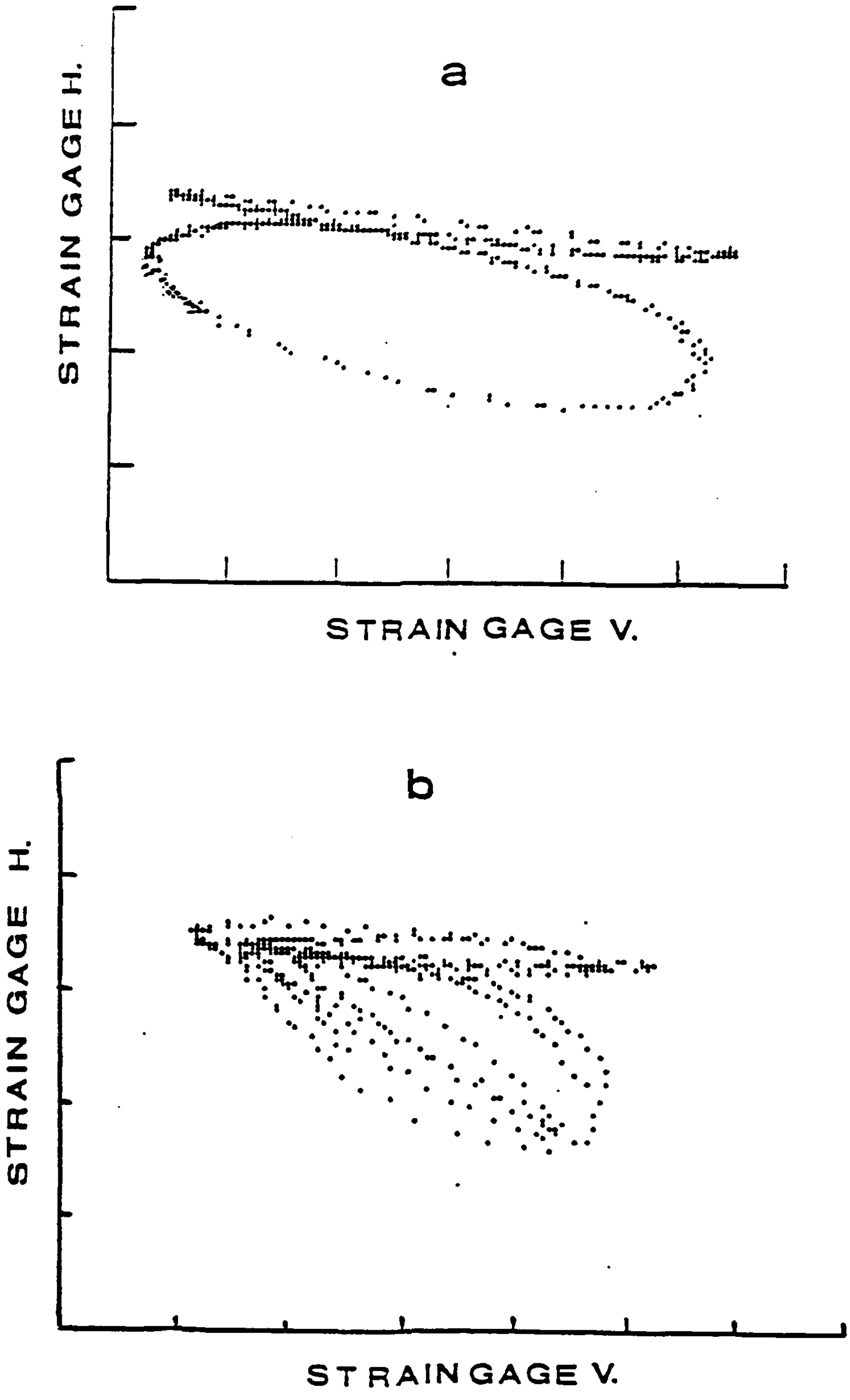


**Figure 25.** Poincáre maps: (a) f = 16.30 Hz, periodically modulated response, and (b) f = 16.325 Hz, chaotically modulated response.